Université de Pau et des Pays de l'Adour

A NEW TOPOLOGICAL INVARIANT OF LINE ARRANGEMENTS

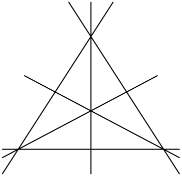
Benoît GUERVILLE-BALLÉ

Joint work :
E. ARTAL BARTOLO
V. FLORENS

Arrangements & topology

Definition

A *complex line arrangement* $\mathcal{A}$ is a set of lines $\{L_0, \cdots, L_n\}$ of $\mathbb{CP}^2$.



Ceva's arrangement (1678)

Definition

The *topological type* of an arrangement $\mathcal{A}$ is the homeomorphism type of the pair $(\mathbb{CP}^2, \bigcup L_i)$.

Combinatorics

Definition

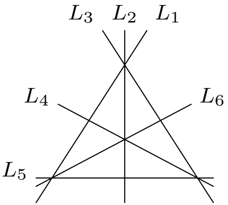
The *combinatorics* of an arrangement $\mathcal{A}$ is a triplet $(\mathcal{A}, \mathcal{Q}, \in)$ formed by :

- the set of lines : $\mathcal{A}$
- the set of intersection points : $\mathcal{Q}$
- the incidence relations between them : $\in$

Example

The combinatorics of Ceva's arrangement is :

$$[\ [1,2,3],[1,4],[1,5,6],[2,4,6],[2,5],[3,4,5],[3,6]\ ]$$



Question

What is the influence of the combinatorics on the topology ?

Question

What is the influence of the combinatorics on the topology ?

Topological invariants :

- Complement : $E_{\mathcal{A}} = \mathbb{CP}^2 \setminus \mathcal{A}$.
- Fundamental group : $\pi_1(E_{\mathcal{A}})$.
- Invariant of the fundamental group :
  - character $\xi \in \mathrm{Hom}(\pi_1(E_{\mathcal{A}}); \mathbb{C}^*)$
  - Alexander's module
  - characteristic varieties

Question

What is the influence of the combinatorics on the topology ?

- (1980) P. Orlik and L. Solomon : *"Combinatorics and topology of complements of hyperplanes"*, (Inventiones Mathematicæ)

- (2001) E. Hironaka : *"Boundary manifold of line arrangements"*, (Mathematische Annalen)

- (1998) G. Rybnikov : "*On the fundamental group of the complement of a complex hyperplane arrangement*",

- (2005) E. Artal, J. Carmona, J.I. Cogolludo and M. Marco : "*Topology and combinatorics of real line arrangements*", (Compositio Mathematica)

Question

What is the influence of the combinatorics on the topology ?

- (1980) P. Orlik and L. Solomon : *"Combinatorics and topology of complements of hyperplanes"*, (Inventiones Mathematicæ)

- (2001) E. Hironaka : *"Boundary manifold of line arrangements"*, (Mathematische Annalen)

- (1998) G. Rybnikov : "*On the fundamental group of the complement of a complex hyperplane arrangement*",

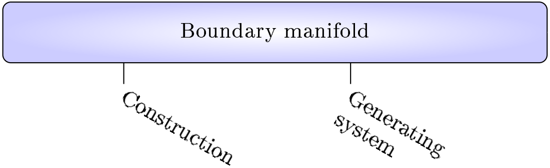
- (2005) E. Artal, J. Carmona, J.I. Cogolludo and M. Marco : "*Topology and combinatorics of real line arrangements*", (Compositio Mathematica)

Definition

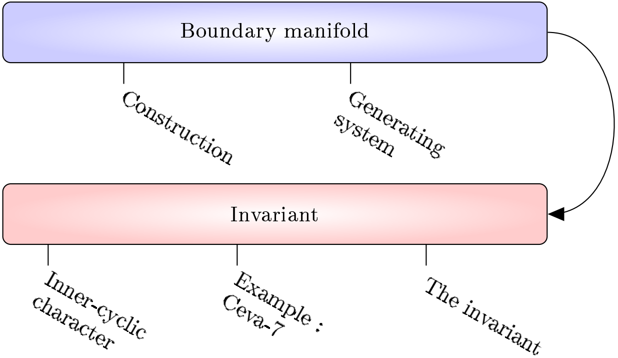
A *Zariski pair* is a pair of arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ with :

- the same combinatorics,
- different topological type.

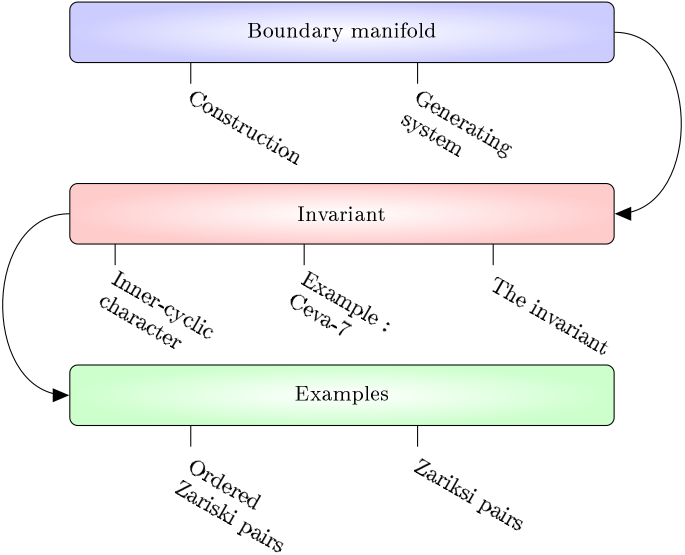
Plan



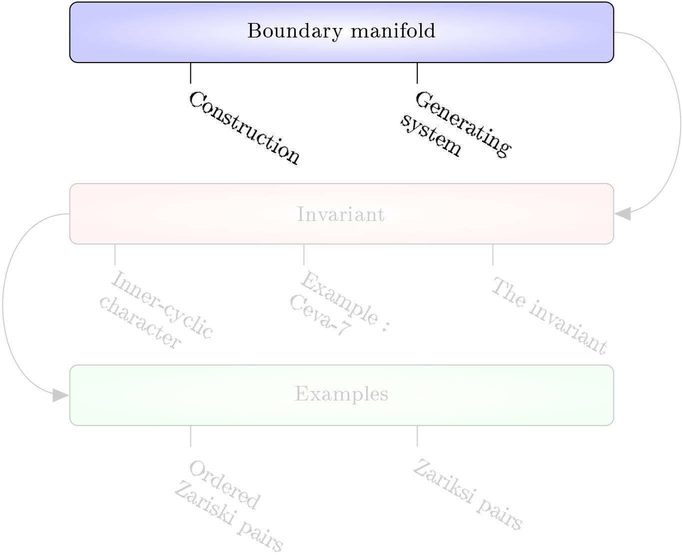
Plan


Boundary manifold
Construction
Generating system
Invariant
Inner-cyclic character
Example : Ceva-7
The invariant

Plan


Boundary manifold
Construction
Generating system
Invariant
Inner-cyclic character
Example : Ceva-7
The invariant
Examples
Ordered Zariski pairs
Zariski pairs

Part 1


Boundary manifold
Construction
Generating system
Invariant
Inner-cyclic character
Example : Ceva-7
The invariant
Examples
Ordered Zariski pairs
Zariksi pairs

Incidence graph

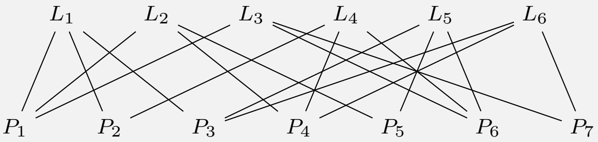
Definition

The incidence graph $\Gamma_{\mathcal{A}}$ of an arrangement $\mathcal{A}$ is a non-oriented bipartite graph where the set of vertices decomposes as $V_P(\mathcal{A}) \sqcup V_L(\mathcal{A})$, with :

$$V_P(\mathcal{A}) = \{v_P \mid P \in \mathcal{Q}\}, \text{ and } V_L(\mathcal{A}) = \{v_L \mid L \in \mathcal{A}\}.$$

Example

The incidence graph of Ceva's arrangement is :



$$[\ [1,2,3], [1,4], [1,5,6], [2,4,6], [2,5], [3,4,5], [3,6]\ ]$$

Construction of the boundary manifold

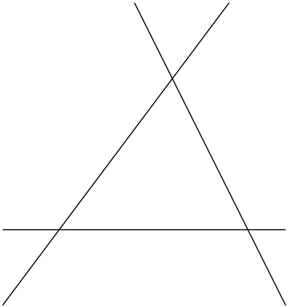
Definition

The *boundary manifold* $B_{\mathcal{A}}$ of an arrangement $\mathcal{A}$ is the boundary of a regular neighborhood of $\mathcal{A}$. We have the inclusion :

$$i : B_{\mathcal{A}} \hookrightarrow E_{\mathcal{A}}.$$

Proposition

The boundary manifold $B_{\mathcal{A}}$ is a graph manifold –in the sense of F. Waldhausen– over the incidence graph $\Gamma_{\mathcal{A}}$.



Construction of the boundary manifold

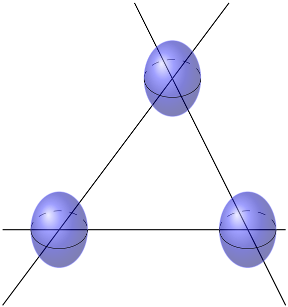
Definition

The *boundary manifold* $B_{\mathcal{A}}$ of an arrangement $\mathcal{A}$ is the boundary of a regular neighborhood of $\mathcal{A}$. We have the inclusion :

$$i : B_{\mathcal{A}} \hookrightarrow E_{\mathcal{A}}.$$

Proposition

The boundary manifold $B_{\mathcal{A}}$ is a graph manifold –in the sense of F. Waldhausen– over the incidence graph $\Gamma_{\mathcal{A}}$.



Construction of the boundary manifold

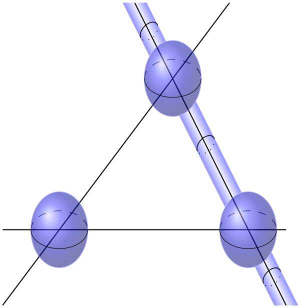
Definition

The *boundary manifold* $B_{\mathcal{A}}$ of an arrangement $\mathcal{A}$ is the boundary of a regular neighborhood of $\mathcal{A}$. We have the inclusion :

$$i : B_{\mathcal{A}} \hookrightarrow E_{\mathcal{A}}.$$

Proposition

The boundary manifold $B_{\mathcal{A}}$ is a graph manifold –in the sense of F. Waldhausen– over the incidence graph $\Gamma_{\mathcal{A}}$.



Construction of the boundary manifold

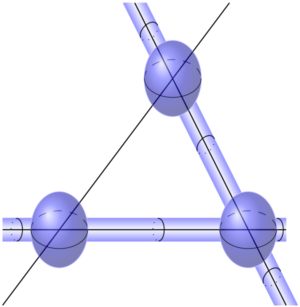
Definition

The *boundary manifold* $B_{\mathcal{A}}$ of an arrangement $\mathcal{A}$ is the boundary of a regular neighborhood of $\mathcal{A}$. We have the inclusion :

$$i : B_{\mathcal{A}} \hookrightarrow E_{\mathcal{A}}.$$

Proposition

The boundary manifold $B_{\mathcal{A}}$ is a graph manifold –in the sense of F. Waldhausen– over the incidence graph $\Gamma_{\mathcal{A}}$.



Construction of the boundary manifold

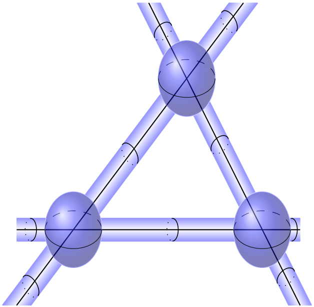
Definition

The *boundary manifold* $B_{\mathcal{A}}$ of an arrangement $\mathcal{A}$ is the boundary of a regular neighborhood of $\mathcal{A}$. We have the inclusion :

$$i : B_{\mathcal{A}} \hookrightarrow E_{\mathcal{A}}.$$

Proposition

The boundary manifold $B_{\mathcal{A}}$ is a graph manifold –in the sense of F. Waldhausen– over the incidence graph $\Gamma_{\mathcal{A}}$.



Generating system of $\pi_1(B_\mathcal{A})$

Proposition

The fundamental group of the boundary manifold is generated by :

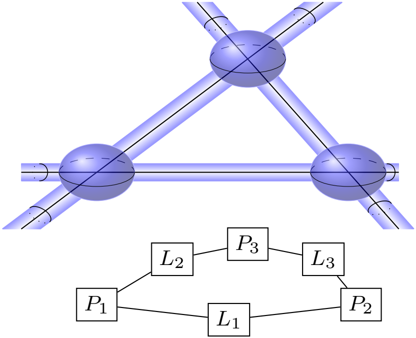
- Meridians around L_i : α_i
- Lift of a basis of cycles of $\Gamma_\mathcal{A}$: $\widetilde{\gamma}_s$

Generating system of $\pi_1(B_{\mathcal{A}})$

Proposition

The fundamental group of the boundary manifold is generated by :

- Meridians around L_i : α_i
- Lift of a basis of cycles of $\Gamma_{\mathcal{A}}$: $\widetilde{\gamma}_s$

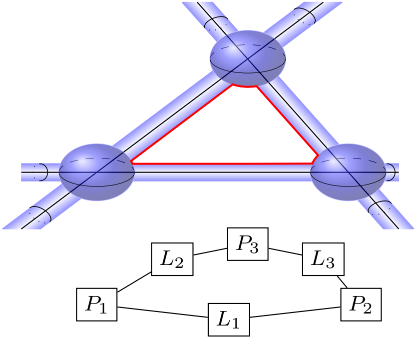


Generating system of $\pi_1(B_{\mathcal{A}})$

Proposition

The fundamental group of the boundary manifold is generated by :

- Meridians around L_i : α_i
- Lift of a basis of cycles of $\Gamma_{\mathcal{A}}$: $\widetilde{\gamma}_s$

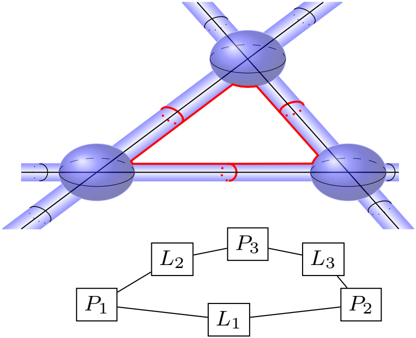


Generating system of $\pi_1(B_{\mathcal{A}})$

Proposition

The fundamental group of the boundary manifold is generated by :

- Meridians around L_i : α_i
- Lift of a basis of cycles of $\Gamma_{\mathcal{A}}$: $\widetilde{\gamma}_s$



Generating system of $\pi_1(B_{\mathcal{A}})$

Proposition

The fundamental group of the boundary manifold is generated by :

- Meridians around L_i : α_i
- Lift of a basis of cycles of $\Gamma_{\mathcal{A}}$: $\widetilde{\gamma}_s$

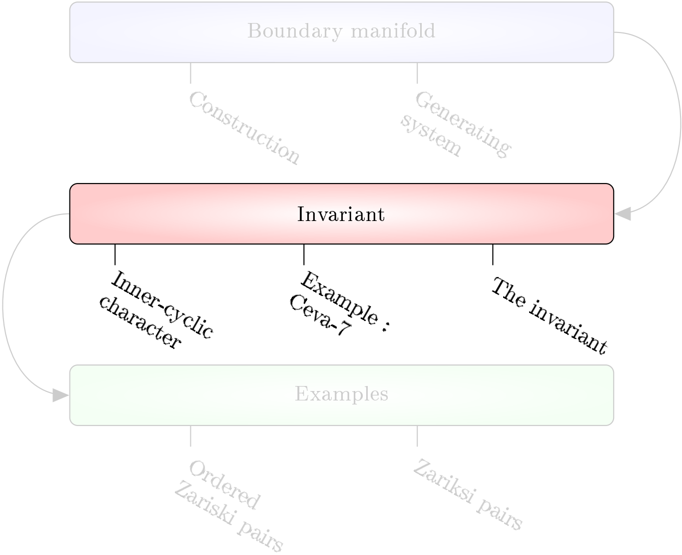
Definition

A lift $\widetilde{\gamma} \in B_{\mathcal{A}}$ of a cycle $\gamma \in \Gamma_{\mathcal{A}}$ is a *nearby cycle* if :

$$\widetilde{\gamma} \in \left(\bigcup_{v_L \in \gamma} T^{\text{hole}}(L) \right) \cup \left(\bigcup_{v_P \in \gamma} \mathbb{B}_P \right),$$

where $T^{\text{hole}}(L)$ is a tubular neighborhood of $L \setminus \left(\bigcup_{P \in L} \mathbb{B}_P \cap L \right)$.

Part 2


Boundary manifold
Construction
Generating system
Invariant
Inner-cyclic character
Example : Ceva-7
The invariant
Examples
Ordered Zariski pairs
Zariksi pairs

Inner cylic character

Let ξ be a character on the complement of $\mathcal{A}$:

$$\xi : \pi_1(E_{\mathcal{A}}) \longrightarrow \mathbb{C}^*.$$

Definition

A character ξ is inner-cyclic if there is a cycle $\gamma \in \Gamma_{\mathcal{A}}$ such that :

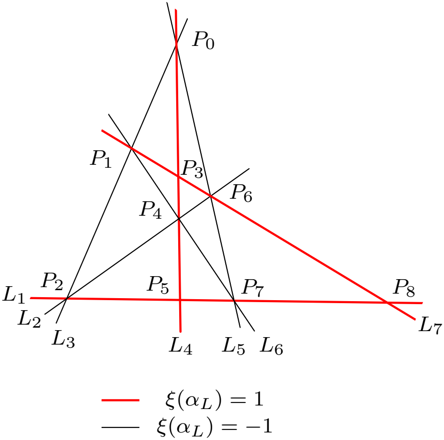
- For all $v_L \in \gamma$, $\xi(\alpha_L) = 1$
- For all $v_P \in \gamma$, $\prod_{L' \ni P} \xi(\alpha_{L'}) = 1$
- For all $P \in L$ with $v_L \in \gamma$, $\prod_{L' \ni P} \xi(\alpha_{L'}) = 1$

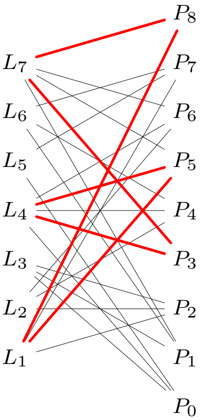
Definition

A inner-cylic arrangement is the data of :

- An arrangement $\mathcal{A}$
- An inner-cyclic character ξ on $\pi_1(E_{\mathcal{A}})$
- A cycle γ associated with ξ

Example : Ceva-7

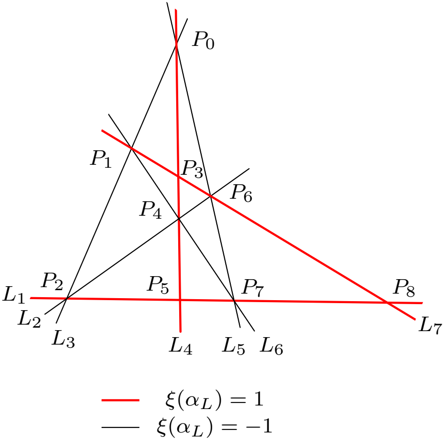


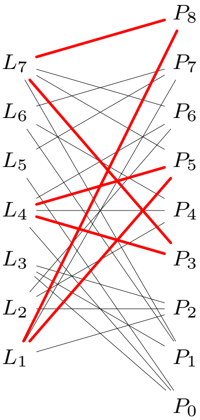


Proposition

The triplet $(\mathcal{A}, \xi, \gamma)$ is an inner-cyclic arrangement.

Example : Ceva-7





Proposition

The triplet $(\mathcal{A}, \xi, \gamma)$ is an inner-cyclic arrangement.

The invariant

Definition

Let $\mathcal{A}$ be an arrangement and ξ a character on $\pi_1(E_\mathcal{A})$. We consider the following application :

$$\chi_{(\mathcal{A},\xi)} : \begin{cases} \Gamma_\mathcal{A} & \xrightarrow{\ell} & B_\mathcal{A} & \xrightarrow{i} & E_\mathcal{A} & \xrightarrow{\xi} & \mathbb{C}^* \\ \gamma & \longmapsto & \tilde{\gamma} & \longmapsto & \tilde{\gamma} & \longmapsto & \xi(\tilde{\gamma}) \end{cases},$$

where ℓ lift any cycle of $\Gamma_\mathcal{A}$ into a nearby cycle.

Theorem (Artal, Florens, ___)

If $(\mathcal{A},\xi,\gamma)$ is an inner-cyclic arrangement, then $\chi_{(\mathcal{A},\xi)}(\gamma)$ is topological invariant of the ordered and oriented pair $(\mathbb{CP}^2,\mathcal{A})$.

E. Artal Bartolo, V. Florens and B. Guerville-Ballé, A topological invariant of line arrangement

The invariant

Definition

Let $\mathcal{A}$ be an arrangement and ξ a character on $\pi_1(E_\mathcal{A})$. We consider the following application :

$$\chi_{(\mathcal{A},\xi)} : \left\{ \begin{array}{ccccccc} \Gamma_\mathcal{A} & \xrightarrow{\ell} & B_\mathcal{A} & \xrightarrow{i} & E_\mathcal{A} & \xrightarrow{\xi} & \mathbb{C}^* \\ \gamma & \longmapsto & \widetilde{\gamma} & \longmapsto & \widetilde{\gamma} & \longmapsto & \xi(\widetilde{\gamma}) \end{array} \right. ,$$

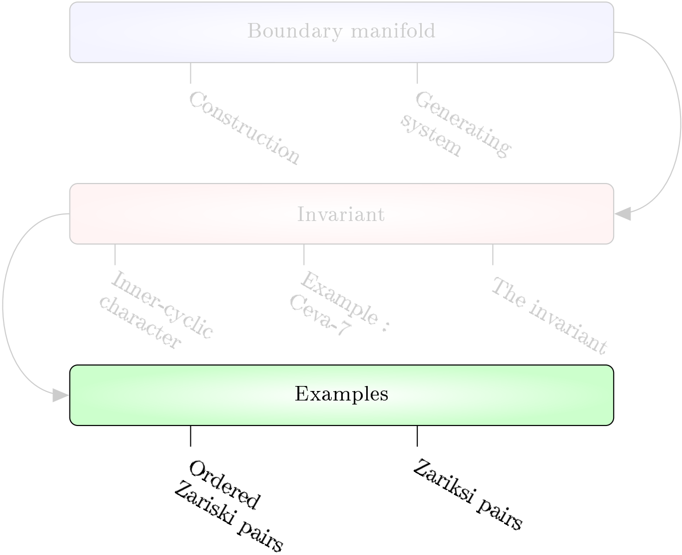
where ℓ lift any cycle of $\Gamma_\mathcal{A}$ into a nearby cycle.

Theorem (Artal, Florens, ___)

If $(\mathcal{A},\xi,\gamma)$ is an inner-cyclic arrangement, then $\chi_{(\mathcal{A},\xi)}(\gamma)$ is topological invariant of the ordered and oriented pair $(\mathbb{CP}^2,\mathcal{A})$.

E. Artal Bartolo, V. Florens and B. Guerville-Ballé, A topological invariant of line arrangement

Part 3


Boundary manifold
Construction
Generating system
Invariant
Inner-cyclic character
Example : Ceva-7
The invariant
Examples
Ordered Zariski pairs
Zariski pairs

Ordered Zariski pair I

Let C be the following combinatorics :

$$C = [\ [1,2], [1,3,5,7], [1,4,6,8], [1,9], [1,10,11], [2,3,6,9],\\ [2,4,5,10], [2,7,11], [2,8], [3,4], [3,8,11], [3,10], [4,7], [4,9,11],\\ [5,6], [5,8,9], [5,11], [6,7,10], [6,11], [7,8], [7,9], [8,10], [9,10]\ ],$$

with the inner-cyclic character :

$$\xi : (\alpha_1, \cdots, \alpha_{11}) \longmapsto (\zeta, \zeta^4, \zeta^3, \zeta^2, 1, 1, \zeta, \zeta^2, \zeta^3, \zeta^4, 1),$$

where ζ is a primitive 5^{th}-root of unity.

Proposition

The combinatorics C admits 4 realizations $\mathcal{A}^+$, $\mathcal{A}^-$, $\mathcal{B}^+$ and $\mathcal{B}^-$:

$$xyz(x-z)(y-z)(x+y-z)(-\alpha^3 x + z)(y - \alpha z)((\alpha-1)x - y + z)\\ (-\alpha(\alpha-1)x + y + \alpha(\alpha-1)z)(-\alpha(\alpha-1)x + y - \alpha z) = 0.$$

with α a roots of the polynomial $X^4 - X^3 + X^2 - X + 1$.

Ordered Zariski pair II

Proposition

The images of the nearby cycles in the complement are :

$$\widetilde{\gamma}_{\mathcal{A}+} = -\alpha_2, \qquad \widetilde{\gamma}_{\mathcal{A}-} = -\alpha_{10} - \alpha_4 - \alpha_8 - \alpha_9,$$
$$\widetilde{\gamma}_{\mathcal{B}+} = -(\alpha_2 + \alpha_9) + \alpha_2, \qquad \widetilde{\gamma}_{\mathcal{B}-} = -(\alpha_2 + \alpha_9).$$

Then, we have :

$$\xi(\widetilde{\gamma}_{\mathcal{A}+}) = \zeta \quad \xi(\widetilde{\gamma}_{\mathcal{A}-}) = \zeta^4 \quad \xi(\widetilde{\gamma}_{\mathcal{B}+}) = \zeta^2 \quad \xi(\widetilde{\gamma}_{\mathcal{B}-}) = \zeta^3$$

Theorem (______)

The pairs $(\mathcal{A}^{\pm}, \mathcal{B}^{\pm})$ are ordered Zariski pairs.

B. Guerville-Ballé, New Zariski pairs of line arrangements.

Ordered Zariski pair II

Proposition

The images of the nearby cycles in the complement are :

$$\widetilde{\gamma}_{\mathcal{A}+} = -\alpha_2, \qquad \widetilde{\gamma}_{\mathcal{A}-} = -\alpha_{10} - \alpha_4 - \alpha_8 - \alpha_9,$$
$$\widetilde{\gamma}_{\mathcal{B}+} = -(\alpha_2 + \alpha_9) + \alpha_2, \qquad \widetilde{\gamma}_{\mathcal{B}-} = -(\alpha_2 + \alpha_9).$$

Then, we have :

$$\xi(\widetilde{\gamma}_{\mathcal{A}+}) = \zeta \quad \xi(\widetilde{\gamma}_{\mathcal{A}-}) = \zeta^4 \quad \xi(\widetilde{\gamma}_{\mathcal{B}+}) = \zeta^2 \quad \xi(\widetilde{\gamma}_{\mathcal{B}-}) = \zeta^3$$

Theorem (____)

The pairs $(\mathcal{A}^{\pm}, \mathcal{B}^{\pm})$ *are ordered Zariski pairs.*

B. Guerville-Ballé, New Zariski pairs of line arrangements.

Zariski pair

Definition

Adding a generic line L_{12} to C passing through $L_1 \cap L_3 \cap L_5 \cap L_7$, we obtain four arrangements denoted by :

$$\mathscr{A}^+, \quad \mathscr{A}^-, \quad \mathscr{B}^+, \quad \mathscr{B}^-.$$

Theorem (____)

The pairs $(\mathscr{A}^\pm, \mathscr{B}^\pm)$ are Zariski pairs.

Theorem (____)

The 4-tuplet $(\mathscr{A}^+, \mathscr{B}^+, \mathscr{A}^-, \mathscr{B}^-)$ is an oriented Zariski 4-tuplet.

Zariski pair

Definition

Adding a generic line L_{12} to C passing through $L_1 \cap L_3 \cap L_5 \cap L_7$, we obtain four arrangements denoted by :

$$\mathscr{A}^+, \quad \mathscr{A}^-, \quad \mathscr{B}^+, \quad \mathscr{B}^-.$$

Theorem (_____)

The pairs $(\mathscr{A}^\pm, \mathscr{B}^\pm)$ are Zariski pairs.

Theorem (_____)

The 4-tuplet $(\mathscr{A}^+, \mathscr{B}^+, \mathscr{A}^-, \mathscr{B}^-)$ is an oriented Zariski 4-tuplet.

Zariski pair

Definition

Adding a generic line L_{12} to C passing through $L_1 \cap L_3 \cap L_5 \cap L_7$, we obtain four arrangements denoted by :

$$\mathscr{A}^+, \quad \mathscr{A}^-, \quad \mathscr{B}^+, \quad \mathscr{B}^-.$$

Theorem (____)

The pairs $(\mathscr{A}^\pm, \mathscr{B}^\pm)$ are Zariski pairs.

Theorem (____)

The 4-tuplet $(\mathscr{A}^+, \mathscr{B}^+, \mathscr{A}^-, \mathscr{B}^-)$ is an oriented Zariski 4-tuplet.

Conclusion

A new topological invariant

- Simply computable
- Detecting Zariski pairs

Conclusion

A new topological invariant

- Simply computable
- Detecting Zariski pairs

Continuations

- Extend the invariant to rational algebraic plane curves
- Describe the inclusion map on the twited homology or on the lower central series

End

Thank you for your attention !