Boundary manifold and complement of complex line arrangement

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Plan

Main objects

$\mathcal{A} := \{L_0, \cdots, L_n\}$ a line arrangement in $\mathbb{CP}^2$.

$E(\mathcal{A}) := \mathbb{CP}^2 - \mathring{T}(\mathcal{A})$ the exterior of $\mathcal{A}$.

$M(\mathcal{A}) := \partial(E(\mathcal{A}))$ the boundary of $E(\mathcal{A})$, also call the boundary manifold.

Hironaka's work

$\pi_1(M(\mathcal{A}))$ is generated by the meridians $\{x_i\}$ around the lines L_i, and the cycles $\{e_i\}$ of the squeleton $\mathcal{A} \cap \mathbb{R}^2$.

The inclusion i of $M(\mathcal{A})$ in $E(\mathcal{A})$ induce an application between the fundamental groups :

$$i_* : \pi_1(M(\mathcal{A})) \longrightarrow \pi_1(E(\mathcal{A})).$$

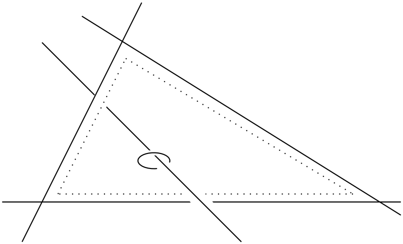
Hironaka's theorem

The kernel of the application i_* is generated by the cycles $\{e_i\}$.

Our contributions

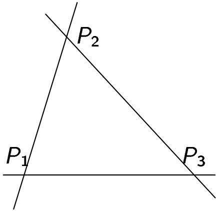
Generalization of the Hironaka's result

The cycles are not contractible in $E(\mathcal{A})$, but they retract on the product of the meridans around the lines passing through the cycles.



Some definitions

Let $\{P_1, \cdots, P_k\}$ the singular points of $\mathcal{A}$,



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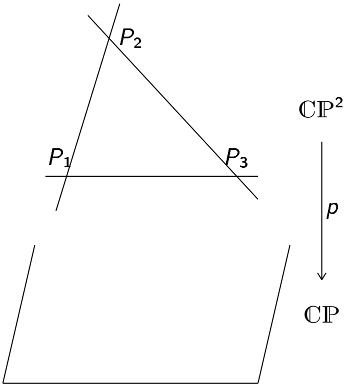

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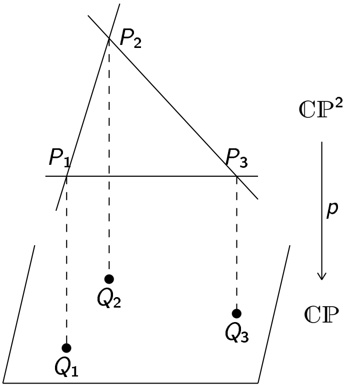

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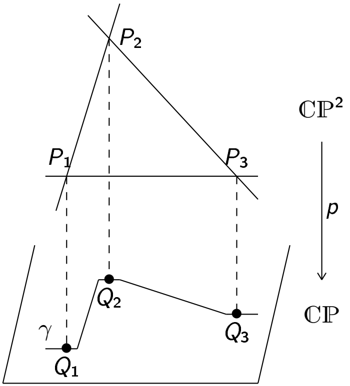


Let γ a path of $\mathbb{CP}$ such that : $\forall i,\ \exists t \in [0,1], \gamma(t) = Q_i$.

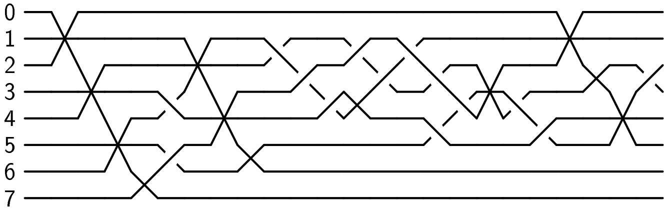
Wiring diagram

definition

The wiring diagram $W_{\gamma,\mathcal{A}}$ is defined by :

$$W_{\gamma,\mathcal{A}} := p^{-1}(\gamma) \cap \mathcal{A}.$$

It is represent by a diagram like :



Fundamental group of the boundary manifold

We know that the boundary manifold is a graph manifold and only depend on the combinatoric of $\mathcal{A}$. So his fundamental group is also combinatoric :

Theorem

The fundamental group $\pi_1(M(\mathcal{A}), b)$ admits the following presentation :

- One set of generators $\{x_i | L_i \in \mathcal{A} - L_0\}$, that represent the loops around the lines.
- One set of generators $\{e_{i,j}\}$, indexed by the loops of the wiring diagram.
- For each singular point P_i, a set of relations given by the cyclic commutator $R_i := [l_{j_1} x_{j_1} l_{j_1}^{-1}, \cdots, l_{j_m} x_{j_m} l_{j_m}^{-1}]$, where $L_{j_1}, \cdots, L_{j_m}$ are the lines that pass through P_i, and l_{j_s} is e_{i,j_s} if e_{i,j_s} is in the previous set, and trivial otherwise.

Futhermore, the combinatoric is include in the wiring diagram $W_{\gamma,\mathcal{A}}$.

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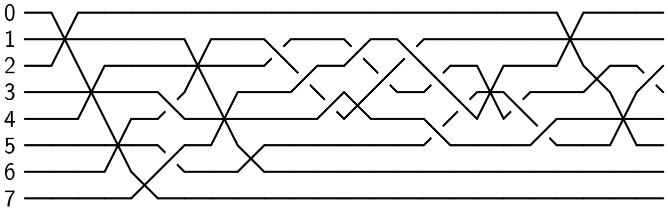
Cycles in the wiring diagram



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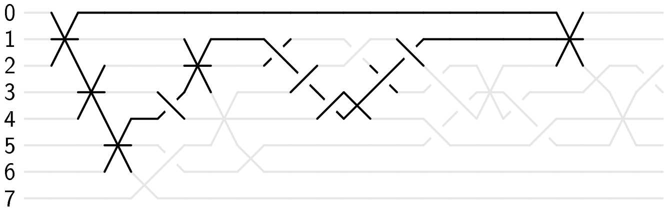

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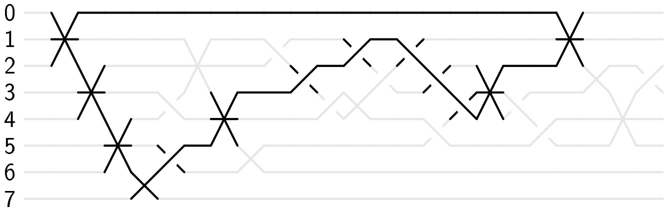

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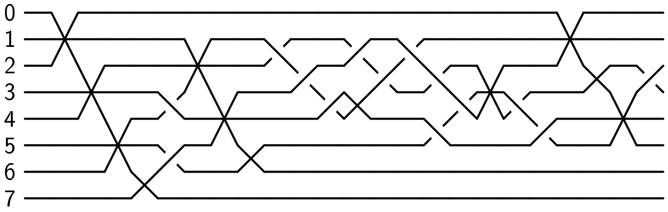


A basis of loop of the wiring diagram is :
$\{e_{2,6},\ e_{2,7},\ e_{1,4},\ e_{1,6},\ e_{4,6},\ e_{3,7}\}$

Uncrossing word

The generators of $\pi_1(M(\mathcal{A}))$ are not exactly the cycles of $W_{\gamma,\mathcal{A}}$. So we need to deinterlace the $e_{i,j}$ for obtain the geometric cycles.

definition

For any cycle ε in $M(\mathcal{A})$, we define the uncrossing word δ_ε as a product of the x_i, such that $\delta_\varepsilon \varepsilon$ is the geometric cycles.

An algorithm to compute the uncrossing word is given in the article of E. Hironaka.

Upper word

Upper segments

Let γ be a cycle of $W_{\mathcal{A}}$ (view as a CW-complex), a segment $\mathfrak{s}$ of $W_{\mathcal{A}}$ intersect uppermost γ if there exists a segment $\mathfrak{s}'$ include in γ such that $\mathfrak{s}$ and $\mathfrak{s}'$ form a virtual crossing of $W_{\mathcal{A}}$ with $\mathfrak{s}$ upper $\mathfrak{s}'$. The set of all the upper segment of γ is noted S_{γ}.

Upper word

For any cycle γ in $W_{\mathcal{A}}$, we define the upper word σ_{γ} by :

$$\sigma_{\gamma} = \prod_{\mathfrak{s} \in S_{\gamma}} a_{\mathfrak{s}}^{e(\mathfrak{s},\gamma)},$$

where $e(\mathfrak{s},\gamma)$ is 1 (resp. -1) if the cross is positive (resp. negative), and $a_{\mathfrak{s}}$ the Arvola's word associated to the segment $\mathfrak{s}$.

Main result

Theorem

The fundamental group of $E(\mathcal{A})$ admit the following presentation :

$$\pi_1(E(\mathcal{A})) = < x_1, \cdots, x_n, \varepsilon_1, \cdots, \varepsilon_k \mid R_i,\ \varepsilon_j = \delta_{\varepsilon_j}^{-1} \sigma_{\varepsilon_j} >$$

In the case of complexified real arrangement, the σ_{ε_j} are trivial, and we obtain E. Hironaka's result.

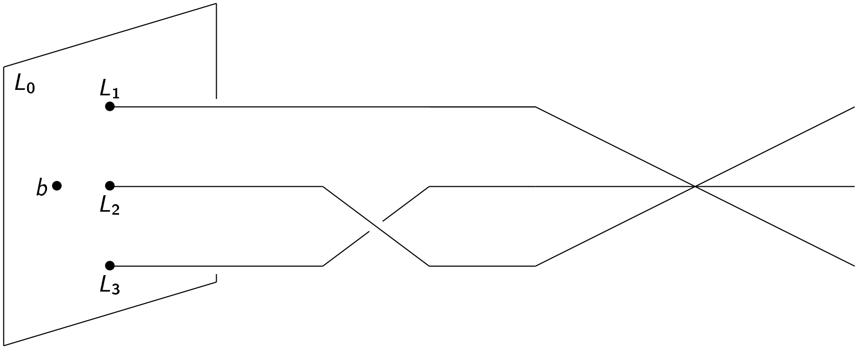
Reformulation of the theorem

Let $\mathcal{A}$ be a complex line arrangement, $M(\mathcal{A})$ be the boundary manifold, and S, the normal sub-group of $\pi_1(M(\mathcal{A}))$ generate by the $\left\{ \delta_\varepsilon \varepsilon \sigma_\varepsilon^{-1} \mid \varepsilon \textit{ cycle of } W_{\gamma,\mathcal{A}} \right\}$. Then we have the following short exact sequence.

$$0 \to S \xrightarrow{\phi} \pi_1(M(\mathcal{A})) \xrightarrow{i_*} \pi_1(E(\mathcal{A})) \to 0,$$

where i_* is induced by the inclusion of $M(\mathcal{A})$ in $E(\mathcal{A})$.

Relations between two syzygys



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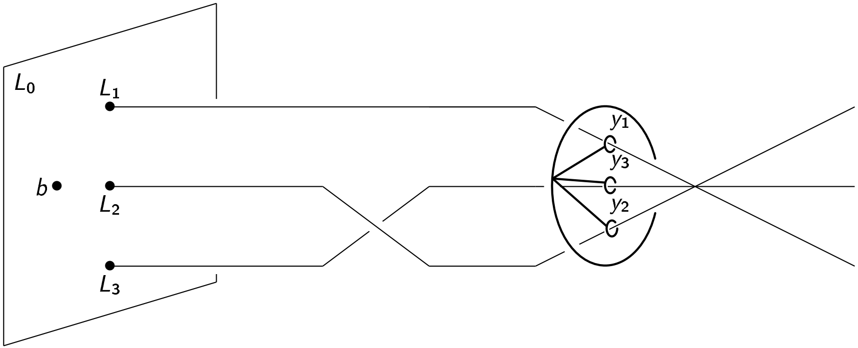

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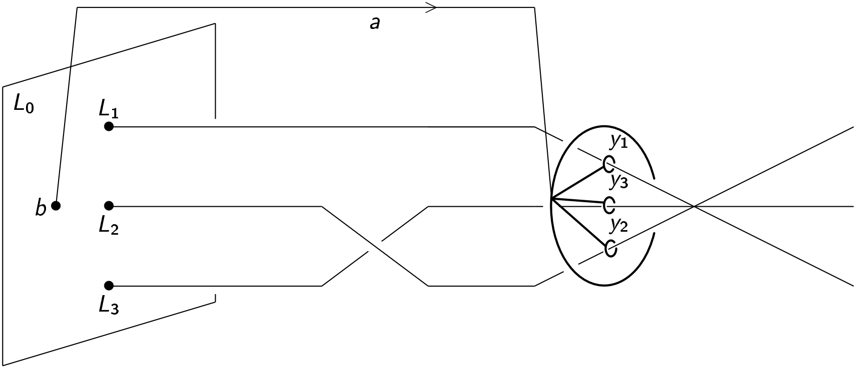

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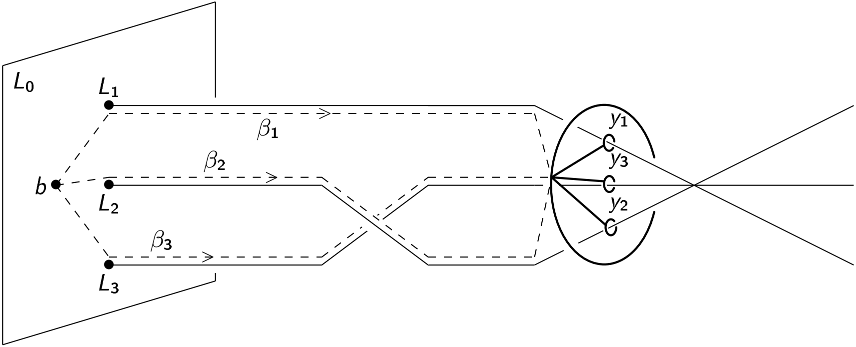

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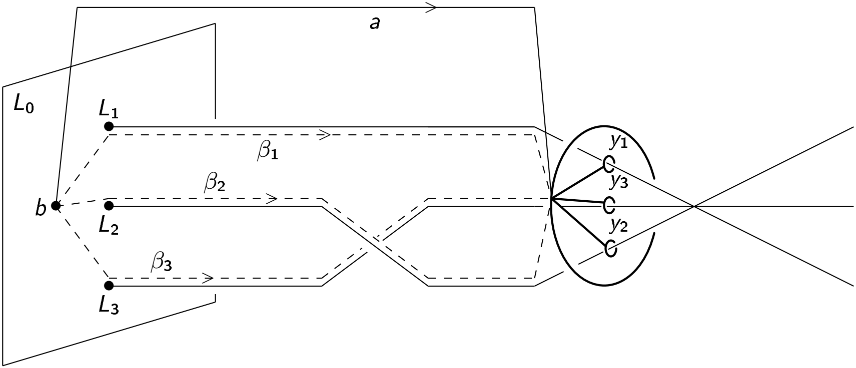

The End

Thank you for your attention.