Boundary manifold and complement of complex line arrangement

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Introduction

Let $\boldsymbol{\mathcal{A} = \{L_0, \cdots, L_n\}}$ be a line arrangement in $\mathbb{P}^2 := \mathbb{CP}^2$. There are two important topological objects associated to $\mathcal{A}$: the pair $(\mathbb{P}^2, \bigcup \mathcal{A})$ and the complement $\mathbb{P}^2 \setminus \bigcup \mathcal{A}$. Let $N(\mathcal{A})$ be a closed regular neigbourhood of $\mathcal{A}$; **the exterior $\boldsymbol{E(\mathcal{A})}$** of $\mathcal{A}$ is the closure $\mathbb{P}^2 \setminus N(\mathcal{A})$. The **boundary manifold $\boldsymbol{M(\mathcal{A})}$** of $\mathcal{A}$ is the common boundary of $E(\mathcal{A})$ and $N(\mathcal{A})$.

The starting point is E. Hironaka's article [Hir97], where she studies the relation between the homotopy type of a real arrangement exterior and the boundary manifold. From this relation she deduces a relation between the fundamental groups of this two spaces. We extend the result to the complex arrangement.

Combinatorics and wiring diagram

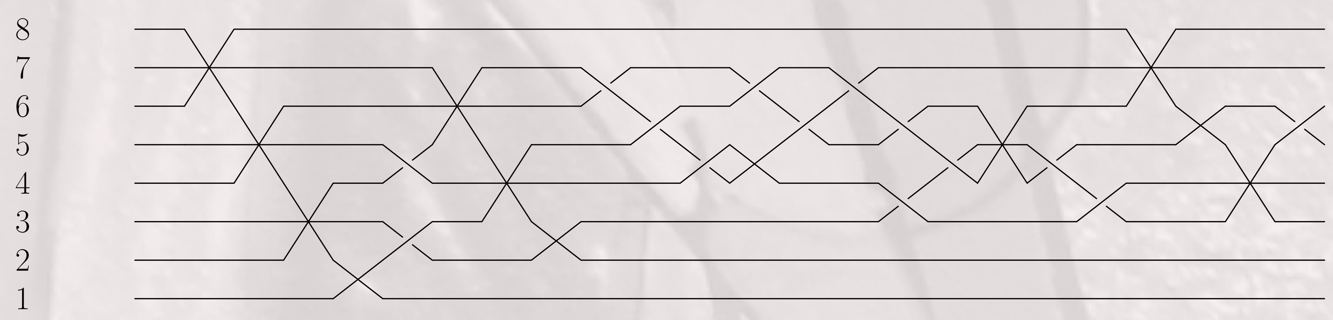
Let $\mathcal{P} = \{P_1, \cdots, P_k\}$ be the set of the singular points of $\mathcal{A}$, $p : \mathbb{C}^2 \to \mathbb{C}$ be a generic projection. We note $\mathcal{Q} = \{Q_1, \cdots, Q_k\}$ the images of the singular points of $\mathcal{A}$ by the projection p.

Consider a path $\gamma : [0,1] \to \mathbb{C}$ with no self-intersection, and such that $\mathcal{Q} \subset \gamma([0,1])$.

Definition. *The wiring diagram associated to the path γ is the subset of $[0,1] \times \mathbb{C}$ defined by :*

$$W_{\mathcal{A},\gamma} = \left\{ (t, p^{-1}(\gamma(t)) \cap \mathcal{A}) \mid t \in [0,1] \right\}$$

For example, the wiring diagram of the positive MacLane arrangement is :



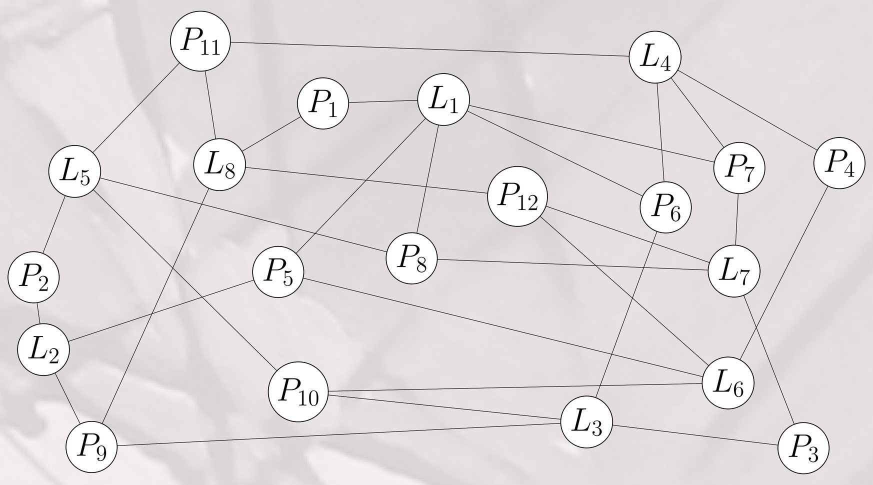
The incidence graph is a subgraph of the Hasse diagram of the arrangement, in which we only keep the vertices of rank 1 and 0. It contain all the combinatorics informations of the arrangement.

Definition. *Let $\mathcal{A}$ be a line arrangement in $\mathbb{CP}^2$, and $\Gamma(\mathcal{A})$ be the non-oriented bipartite graph defined by :*

$$\begin{array}{l} Point-vertices : v_P,\ P \in \mathcal{P} \\ Line-vertices\ : v_L,\ L \in \mathcal{A}. \end{array}$$

The edges of $\Gamma(\mathcal{A})$ are of the form $Y(L,P)$, with $P \in \mathcal{P}$, $L \in \mathcal{A}$ and $P \in L$.

The Figure below gives the incidence graph of the positive MacLane arrangement.



The boundary manifold

The boundary manifold depends only on the combinatorics of $\mathcal{A}$. The following proposition describes a presentation of $\pi_1(M(A))$ from the incidence graph.

Proposition ([BGB12])**.** *The fundamental group $\pi_1(M(\mathcal{A}))$ admits the following presentation:*

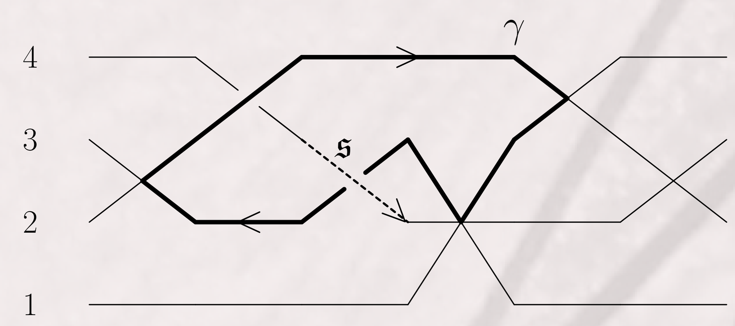
- *A set of generators $\{x_i \mid L_i \in \mathcal{A}\}$, that represent the loops around the lines.*
- *A set of generators $\{e_{i,j}\}$, indexed by the edges $Y(P_i, L_j)$ that are not in the maximal tree.*
- *For each singular point P_i, a set of relations given by the cyclic commutator $[l_{j_1} x_{j_1} l_{j_1}^{-1}, \ldots, l_{j_m} x_{j_m} l_{j_m}^{-1}]$ where $L_{j_1}, \ldots, L_{j_m}$ are the lines that pass through P_i, and l_{j_s} is e_{i,j_s} if $Y(P_i, L_{j_s})$ is not in the maximal tree, and trivial otherwise.*

The main result

Definition. *For any cycle γ in $W_{\mathcal{A}}$, we define the upper word σ_γ by :*

$$\sigma_\gamma = \prod_{\mathfrak{s} \in S_\gamma} a_{\mathfrak{s}}^{e(\mathfrak{s},\gamma)},$$

where $e(\mathfrak{s},\gamma)$ is 1 (resp. -1) if the crossing is positive (resp. negative), and $a_{\mathfrak{s}}$ the word of Arvola of $\mathfrak{s}$ and S_γ the set of segment of $W_{\mathcal{A}}$ intersecting uppermost γ.



In the example at the left, the segment $\mathfrak{s}$ (dashed line) is the only one upper segment of the cycle γ. The Arvola's word of $\mathfrak{s}$ is $a_{\mathfrak{s}} = x_2 x_4 x_2^{-1}$. So we obtain that :

$$\sigma_\gamma = x_2 x_4^{-1} x_2^{-1},$$

because the crossing between γ and $\mathfrak{s}$ is negative.

Definition. *For any cycle ε in $M(\mathcal{A})$, we define the uncrossing word δ_ε as a product of the x_i such that $\delta_\varepsilon \varepsilon$ is the path in $Im(\sigma)$ corresponding to ε (i.e. such that $\forall e \in \pi_1(\Gamma_{\mathcal{A}}), e \in \sigma^{-1}(\varepsilon) \Rightarrow \sigma(e) = \delta_\varepsilon \varepsilon$).*

Let S be the normal sub-group of $\pi_1(M(\mathcal{A}))$ generated by the elements $\delta_\varepsilon \varepsilon \sigma_\varepsilon^{-1}$, where ε are the cycles of $\pi_1(M(\mathcal{A}))$.

Theorem ([BGB12])**.** *Let $\mathcal{A}$ be a complex line arrangement, $M_{\mathcal{A}}$ be the boundary manifold, and $\Gamma_{\mathcal{A}}$ the incidence graph of $\mathcal{A}$. There exists a group S such that the following short sequence is exact.*

$$0 \to S \xrightarrow{\phi} \pi_1(M(\mathcal{A})) \xrightarrow{i_*} \pi_1(E(\mathcal{A})) \to 0,$$

where i_* is induced by the inclusion of $M(\mathcal{A})$ in $E(\mathcal{A})$.
Furthermore, a presentation of S can be computed from the wiring diagram $W_{\mathcal{A},\gamma}$.

Moreover, the generators of this presentation of S can be expressed in terms of the generators of Proposition below.

Sketch of the proof :

- The surjectivity of i_* comes from the Zariski-Van Kampen and the isomorphism between $\pi_1(\mathbb{C}\mathcal{P}^2 - \mathcal{A})$ and $\pi_1(\mathbb{C}^2 - (\mathcal{A} - L_0))$.
- Since S is constructed as a subgroup of $\pi_1(M(\mathcal{A}))$, then Φ is one-to-one.
- We glue on the cycle of the boundary manifold some 2-cells which give the retractation of $\delta_\varepsilon \varepsilon$ on σ_ε. So the composition map $i_* \circ \Phi$ is zero.
- To show the exactness of the short exact sequence, we prove that the relation of the quotient $\pi_1(M(\mathcal{A}))/S$ implies the usual relation of $\pi_1(E(\mathcal{A}))$.

References

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