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A stroll through the topology of line arrangements and their moduli spaces.

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"*Keep searching for what is possible.*"

Courtney Dauwalter

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Contents

1. First steps on the trail

A *line arrangement* is a finite set $\mathcal{A} = \{\ell_1, \dots, \ell_n\}$ of complex projective lines in $\mathbb{CP}^2$. Each line ℓ_i can be defined as the zero locus of a 1-form α_i. The *defining polynomial* of $\mathcal{A}$ is then the product $Q(\mathcal{A}) = \prod_{i=1}^n \alpha_i$. The vanishing points of $Q(\mathcal{A})$ are the *zero loci* $\mathcal{Z}(\mathcal{A})$ of the arrangement $\mathcal{A}$, and it can be defined as the union of the lines of $\mathcal{A}$, i.e. $\mathcal{Z}(\mathcal{A}) = \bigcup_{i=1}^n \ell_i \subset \mathbb{CP}^2$. The *complement* of $\mathcal{A}$ is $M(\mathcal{A}) = \mathbb{CP}^2 \setminus \mathcal{Z}(\mathcal{A})$. The arrangement $\mathcal{A}$ is a *complexified real arrangement*, if there exists a system of coordinates of $\mathbb{CP}^2$ such that each 1-form ℓ_i has an equation with real coefficients[1]. On the first hand, the definition of $\mathcal{A}$ as the vanishing points of the polynomial $Q(\mathcal{A})$ endowed it with a structure of *algebraic plane curve*; on the other hand, $\mathcal{A}$ is also defined as the vanishing locus of 1-forms that makes it a *hyperplane arrangement*. This specific position of line arrangement at the intersection of these two fundamental domains of mathematics provides to the study of line arrangements a myriad of aspects. Our main focus is on the *topology* of line arrangements, nevertheless *combinatorial* or *geometric* aspects will play important roles in this study.

1.1. Origins: Hyperplane arrangements & Algebraic plane curves.

The study of hyperplane arrangements takes its origins in the study of finite reflection groups and braid groups. Indeed, the pure braid space $M = \{x \in \mathbb{C}^l \mid x_i \neq x_j \text{ for } i \neq j\}$ is the complement of a hyperplane arrangement. In 1962, Fadell, Fox, and Neuwirth [30, 39] proved that this space is $K(\pi, 1)$, opening the path to the study of the topology of hyperplane arrangements complement. Then, Arnol'd gave a finite presentation of the cohomology ring $\mathrm{H}^*(M)$ depending only on the intersection lattice of the arrangements [4]. In the same paper, he conjectured that the cohomology ring of the complement of any hyperplane arrangement is torsion-free and generated by some specific one-dimensional classes. In 1971, Brieskorn gave a proof of that fact in a Bourbaki Seminar talk [16]. This result has been extended by Brieskorn and Saito to generalized Artin groups [17]. Independently, Deligne also obtained the same generalization [25]. In addition, he proved that when the associated arrangement is simplicial[2], its complement is an Eilenberg–MacLane space, more precisely it is $K(\pi, 1)$. This solved a conjecture made by Brieskorn the year before during the Bourbaki Seminar. Finally, the original presentation of $\mathrm{H}^1(M)$ obtained by Arnol'd has been generalized in [66] to any hyperplane arrangement by Orlik and Solomon in 1980. In particular, this implies that the cohomology ring of a hyperplane arrangement complement is determined by its intersection lattice. This naturally leads to the following question.

Question. *How much of the embedded topology of a hyperplane arrangement is determined by its intersection lattice?*

In 1925, Enriques questioned the existence of complex algebraic function z of x and y, possessing a preassigned curve f as branch curve [29]. In a private conversation with Zariski, Lefschetz reduced this problem to the determination the fundamental group of the given curve complement. Then, during the 30s', Zariski published a series of three papers [88, 89, 90] in which he proved the existence of two irreducible sextics with six cusps that have non-isomorphic fundamental groups. In both cases, the fundamental group is a finite group of order 6, but Zariski proved that it is non-Abelian if and only if the six cusps lie on a conic. Even if he gave an explicit equation of the curve with a non-Abelian fundamental group, we had to wait fifty-five years and the paper of Oka [63] to have an equation for

[1] Note that this condition is stronger than to assume that $Q(\mathcal{A})$ is a polynomial with real coefficients.

[2] An arrangement is simplicial if it is a complexified real arrangement and its complement in the real plane is the disjoint union of simplicial cones.

the Abelian case[3]. Zariski claimed that only two families of such cuspidal sextics exist. Degtyarev classified in [24] the other families of sextics. In [5], Artal suggested calling such couples of curves with the same local singularities yet different embedded topology a *Zariski pair*. During the end of the last century, various authors produced new examples of Zariski pairs: Artal [5], Oka [64], Shimada [75] and Degtyarev [23]. We refer to the survey [9] by Artal, Cogolludo and Tokunaga for more details about Zariski pairs of algebraic plane curves.

Extending the underlying meaning of *local types of singularities* into the notion of *combinatorics* [5], one can extend the previous definition of Zariski pairs to any algebraic plane curve (not necessarily irreducible). The first example[4] of such reduced Zariski pair has been exhibited by Artal in [5]. It was at this point that this story met the previous one in the form of the following question.

Question. *Does Zariski pairs of line arrangements exist?*

1.2. **Topology of line arrangements.**

The result of Orlik and Solomon [66] suggests that the topology of a line arrangement is determined by its intersection lattice[5]; while Zariski's works hint the converse [88, 89, 90].

1.2.1. *Homotopy of the complement.*

The first step to tackle this problem was to determine the fundamental group of a line arrangement. The first method to compute a finite presentation is due to Zariski and van Kampen [88, 80] in the general context of algebraic curves. In [20], Chissini noticed that this presentation involves a powerful invariant: the *braid monodromy* (see also [19, 60]). In the particular case of complexified real arrangements, Randell gave an algorithm based on the real picture of the arrangement to compute the fundamental group [68, 69, 73]. Then, it has been adapted to any line arrangement by Arvola [14] using the so-called *braided wiring diagram*. It is a diagrammatic method to encode the braid monodromy of a line arrangement. Recently, Yoshinaga gave another presentation in the particular case of complexified real arrangements [87]. We can also mention the presentation obtained in [38] using the inclusion of the boundary manifold[6] in the complement. Several examples of fundamental groups and various invariants have been computed by Suciu in [78]. In this paper, he noticed that the lower central series quotients may have some torsion, and he queries whether it is determined by the intersection lattice or not.

In a wider perspective, mathematicians are also interested by the homotopy type of the complement. In [32], Falk constructed two homotopy-equivalent arrangements with the same *weak combinatorics*[7] yet different intersection lattices. In the other direction, Jiang and Yau proved that the homeomorphism type of the complement determines the intersection lattice [52] (see also Di Pasquale [27]). We refer to the survey of Falk and Randell for various open problems on the homotopy of hyperplane arrangements [36].

[3] I didn't wait that long since I was only 6 years old when Oka's paper has been published.

[4] These curves are formed by a smooth cubic and three inflectional tangent lines. Noway, such curves are known as the 3-Artal curves [15].

[5] The intersection lattice is the equivalent for hyperplane arrangements of the combinatorics for algebraic plane curves. In the case of line arrangements, they will be used as synonyms.

[6] The boundary manifold of an arrangement $\mathcal{A}$ is a graph 3-manifold, in the sense of Waldhausen [81], defined as the boundary of a regular tubular neighborhood of $\mathcal{Z}(\mathcal{A})$ in $\mathbb{CP}^2$.

[7] The weak combinatorics is given by the number of lines, the number of singular points for each multiplicity and their repartition among the lines. It differs from the combinatorics, by its lack of incidence relations between the lines.

1.2.2. *Zariski pairs of line arrangements exist!*

It was in 1998 that Rybnikov discovered the first example of a Zariski pair of line arrangements [71]. The Rybnikov arrangements are formed by 13 lines with only double and triple points and are defined over $\mathbb{Q}[\zeta_3]$, with ζ_3 a primitive third root of unity. To distinguish their topology, Rybnikov introduced an invariant of the fundamental group of the arrangement based on its lower central series. The difficulty of Rybnikov's proof made it hard to review, and he almost gave up publishing it. It was finally published in 2011 [72]. Meanwhile, Artal, Carmona, Cogolludo and Marco published a detailed proof of Rybnikov's result [8].

The second example of a Zariski pair appeared in 2005, and it is due to Artal, Carmona, Cogolludo and Marco [7]. Their example is formed by two arrangements of 11 lines with double and triple points together with a unique point of multiplicity 5. They have the particularity to be complexified real arrangements defined over $\mathbb{Q}[\sqrt{5}]$, and their topologies are distinguished using the braid monodromy. It is a fine invariant of the embedded topology which determines the homotopy type of the complement [54]. Nevertheless, this difference between the braid monodromies does not imply a difference between neither the fundamental groups, nor the homeomorphism types of the complements. Until now, we did not know whether they are homeomorphic or not. Therefore, this strengthens our interest for the Falk and Randell problem 1.3 in [36].

Problem. *Is the fundamental group of a complexified real arrangement determined by its intersection lattice?*

1.2.3. *My contributions.*

The contents presented in this manuscript shed fresh insights on the study of line arrangements. The introduction of a linking invariant for line arrangements enables the detection of new Zariski pairs (Sections 2 and 4). Notably, the diversity of these pairs facilitates the resolution of several open questions, such as the combinatorial nature of the fundamental group of complexified real arrangements (Section 5.1) or of Galois-conjugated arrangements (Section 2.3). By combining this invariant with Rybnikov's idea, we achieved to generalize his construction (Section 3). Remarkably, this approach also leads to Zariski pairs that are homotopy-equivalent yet possess non-homeomorphic complements (Section 5.2).

Our exploration of Zariski pairs of line arrangements naturally guides us toward an examination of the topological aspects of the moduli space of line arrangements. Inspired by the pioneering work of Nazir and Yoshinaga, we extended their combinatorial class of C_3 arrangements of simple type (Section 6.5). Subsequently, we established a sharp upper bound for the number of connected components of the moduli space (Section 7).

1.3. **Definitions and notations.**

To avoid confusion and misleading, let us fix some definitions and notations. They will be used across all this manuscript. If different notations or assumptions are made for some statements, they will be explicitly mentioned. The set of all line arrangements with n lines is denoted by Arr_n; furthermore, in this manuscript, n will always denote the cardinality of an arrangement $\mathcal{A}$.

1.3.1. *Topology.*

The main object of our interest in this manuscript is the *topology* of an arrangement $\mathcal{A}$. It is defined as the homeomorphism type of $(\mathbb{CP}^2, \mathcal{Z}(\mathcal{A}))$. A meridian around the line $\ell \in \mathcal{A}$ is an oriented path homotopically equivalent to the boundary of a small disc transverse to ℓ at a smooth point of $\mathcal{Z}(\mathcal{A})$. It is denoted by m_ℓ or $\mathfrak{m}_\ell$. Depending on the context, a meridian can be considered either as a path in $M(\mathcal{A})$, as its homotopy class in $\pi_1(M(\mathcal{A}))$ or as its homology class in $\mathrm{H}_1(M(\mathcal{A}))$. The *oriented topology* of $\mathcal{A}$ is the restriction of the topology to the class of homeomorphisms preserving both the global orientation of $\mathbb{CP}^2$ and the local orientation of the meridians m_ℓ.

An *ordered arrangement* is a pair $(\mathcal{A}, \omega)$, where $\mathcal{A} \in \mathrm{Arr}_n$ and ω is a bijective map $\omega : \mathcal{A} \longrightarrow \{1, \dots, n\}$, called the *order* on $\mathcal{A}$. For the sack of brevity, we often express an ordered arrangement as a set $\mathcal{A} = \{\ell_1, \dots, \ell_n\}$ where the indices of the lines are chosen in a compatible way with the order ω, i.e. such that $\omega(\ell_i) = i$, for any $\ell_i \in \mathcal{A}$. The *ordered topology* of $(\mathcal{A}, \omega)$ is the restriction of the topological type to homeomorphisms respecting the order ω (i.e. to homeomorphisms $h : \mathbb{CP}^2 \to \mathbb{CP}^2$ such $\omega(\ell_i) = \omega \circ h(\ell_i)$ for all $i \in \{1, \dots, n\}$). When ω is the order given by the indices then the ordered topology is the homeomorphism type of the tuple $(\mathbb{CP}^2, \ell_1, \dots, \ell_n)$. In this manuscript, we consider only ordered arrangements (without specifying each time that it is ordered). Exceptions to this rule occur, for example when we obtain Zariski pairs, they are explicitly indicated each time.

1.3.2. *Combinatorics.*

In [5, Remark 1.2], the *combinatorics* of an algebraic plane curve C is defined as the data of the degree of each irreducible component $C_1, \dots, C_n$ of C, the list of the topological type $\Sigma_1, \dots, \Sigma_m$ of the singular points of C, and for each branch Σ_i^j of a singularity Σ_i the irreducible component C_k of C which contains Σ_i^j. Orlik and Terao [67, Definition 1.12] define the *intersection lattice* of a hyperplane arrangement as the set of all non-empty intersections of elements of the arrangement. When $\mathcal{A}$ is a line arrangement then this definition can be reduced to the set of all the maximal subsets $\mathcal{A}_P$ of $\mathcal{A}$ with a non-empty intersection[8]. As already mentioned, in the context of line arrangement these two definitions are equivalent. For convenience in the notation, we will mainly use the reduced version of the intersection lattice; nevertheless, since our main influence comes from the study of algebraic curves, it will be called combinatorics in this manuscript. Explicitely, the combinatorics $\mathcal{C}(\mathcal{A})$ of $\mathcal{A}$ is given by:

$$\mathcal{C}(\mathcal{A}) = \left\{ \mathcal{A}_P \subset \mathcal{A} \;\middle|\; \bigcap_{\ell_i \in \mathcal{A}_P} \ell_i \neq \emptyset \;\text{ and }\; \forall \ell \in \mathcal{A} \setminus \mathcal{A}_P :\; \ell \cap \bigcap_{\ell_i \in \mathcal{A}_P} \ell_i = \emptyset \right\}.$$

Frequently, $\mathcal{A}_P$ will be identified as the singular point $P \in \mathrm{Sing}(\mathcal{A})$ of $\mathcal{Z}(\mathcal{A})$. The *multiplicity* of a singular point P is $m(P) = |\mathcal{A}_P|$, it is a *multiple point* if $m(P) \geq 3$.

Two arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ are combinatorially equivalent, if there exists a one-to-one set-correspondence ϕ between $\mathcal{A}_1$ and $\mathcal{A}_2$ such that for all $\mathcal{A}_P \in \mathcal{C}(\mathcal{A})$, one has $\phi(\mathcal{A}_P) \in \mathcal{C}(\mathcal{A}_2)$. This equivalence is written $\mathcal{C}(\mathcal{A}_1) \sim \mathcal{C}(\mathcal{A}_2)$. When $\mathcal{A}_1$ and $\mathcal{A}_2$ are two ordered arrangements, with respective orders ω_1 and ω_2, then ϕ needs, in addition, to respect the orders, i.e. $\omega_1(\ell_i) = \omega_2 \circ \phi(\ell_i)$ for all $i \in \{1, \dots, n\}$.

The *automorphism group* of a combinatorics $\mathcal{C}(\mathcal{A})$ is the subgroup of the permutation group $\Sigma(\mathcal{A})$ which fixes $\mathcal{C}(\mathcal{A})$. More precisely, one has:

$$\mathrm{Aut}(\mathcal{C}(\mathcal{A})) = \{\sigma \in \Sigma(\mathcal{A}) \mid \forall \mathcal{A}_P \in \mathcal{C}(\mathcal{A}), \sigma(\mathcal{A}_P) \in \mathcal{C}(\mathcal{A})\}.$$

An *abstract line combinatorics* is a combinatorial structure that mimics the combinatorics of a line arrangement. It is defined as $\mathcal{C} = (\mathcal{L}, \mathcal{P})$ such that $\mathcal{P}$ is a subset of the powerset of $\mathcal{L}$ and it verifies

(1) for all P in $\mathcal{P}$, $|P| \geq 2$,
(2) for all $\ell_1 \neq \ell_2 \in \mathcal{L}$, there exists a unique $P \in \mathcal{P}$ such that $\ell_1 \in \mathcal{P}$ and $\ell_2 \in P$.

[8] In this context, maximal means that for any line $\ell \in \mathcal{A} \setminus \mathcal{A}_P$, the intersection $\ell \cap \left(\bigcap_{\ell_i \in \mathcal{A}_P} \ell_i\right)$ is empty.

If $\mathcal{A}$ is a line arrangement then $\mathcal{C}(\mathcal{A})$ is an abstract line combinatorics. Nevertheless, the converse is not true in general. Indeed, using Pappus' hexagon theorem, one can construct an abstract line combinatorics which is not the combinatorics of a line arrangement.

1.3.3. *Moduli space.*

The *realization space* $\mathcal{R}(\mathcal{A})$ of an ordered arrangement $\mathcal{A}$ (or equivalently of $\mathcal{C}(\mathcal{A})$) is the subset $\mathcal{R}(\mathcal{A})$ of Arr_n defined by:

$$\mathcal{R}(\mathcal{A}) = \{\mathcal{B} \in \mathrm{Arr} \mid C(\mathcal{A}) \sim \mathcal{C}(\mathcal{B})\}.$$

A line $\ell_i : a_i x + b_i y + c_i z = 0$ in $\mathbb{CP}^2$ can be considered as a point $(a_i : b_i : c_i)$ in the dual $\check{\mathbb{CP}^2}$, then an element of Arr_n is identified with a point in $\left(\check{\mathbb{CP}^2}\right)^n$. Three distinct lines ℓ_i, ℓ_j and ℓ_k are *concurrent* if and only if

$$\Delta_{i,j,k} := \det(\ell_i, \ell_j, \ell_k) = \begin{vmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{vmatrix} = 0.$$

By definition, the realization space of a line combinatorics $\mathcal{C}(\mathcal{A})$ with $\mathcal{A} \in \mathrm{Arr}_n$ can be constructed as a subset of $\left(\check{\mathbb{CP}^2}\right)^n$ as follows:

$$\mathcal{R}(\mathcal{A}) = \left\{ (\ell_1, \dots, \ell_n) \in \left(\check{\mathbb{CP}^2}\right)^n \;\middle|\; \begin{array}{ll} \ell_i \neq \ell_j, & \forall i \neq j \\ \Delta_{i,j,k} = 0, & \text{if } \{\ell_i, \ell_j, \ell_k\} \subset \mathcal{A}_P, \text{ for a } \mathcal{A}_P \in \mathcal{C}(\mathcal{A}) \\ \Delta_{i,j,k} \neq 0, & \text{otherwise} \end{array} \right\}.$$

There is a natural action of $\mathrm{PGL}_3(\mathbb{C})$ on $\mathcal{R}(\mathcal{A})$. Indeed, any projective transformation preserves lines intersection, so if $T \in \mathrm{PGL}_3(\mathbb{C})$, then the arrangements $T \cdot \mathcal{A} = \{T(\ell_1), \dots, T(\ell_n)\}$ and $\mathcal{A}$ are combinatorially equivalent. That is to say, $T \cdot \mathcal{A} \in \mathcal{R}(\mathcal{A})$. We define thus the *moduli space* $\mathcal{M}(\mathcal{A})$ of $\mathcal{A}$ (or of $\mathcal{C}(\mathcal{A})$) as the quotient of $\mathcal{R}(\mathcal{A})$ by this action of $\mathrm{PGL}_3(\mathbb{C})$. It is worth noticing that the realization space and moduli space that we consider here correspond respectively to the ordered realization space $\Sigma^{\mathrm{ord}}(\mathcal{C})$ and ordered moduli space $\mathcal{M}^{\mathrm{ord}}(\mathcal{C})$ in [7, Definition 3.3].

As proved by Mnëv in [59], the moduli space of a line arrangement can behave as badly as one can imagine. Vakil qualifies such behavior as *Murphy's law* [79]. More precisely, Mnëv Universality Theorem states that every singularity of finite type over $\mathbb{Z}$ appears in at least one moduli space. Recently, it was shown in [22] that the realization space of line arrangements is smooth for $|\mathcal{A}| \leq 11$, and then nodal singularities appear for $|\mathcal{A}| = 12$. In addition, there are several basic geometric or topological aspects of the moduli space that cannot be fully predicted by a combinatorial study, as shown by the classical Pappus' hexagon theorem.

2. An adaptation of the linking number to line arrangements

In this first section, we present a topological invariant of line arrangements named the $\mathcal{I}$-*invariant*[9]. It is inspired by the linking number of Knot Theory and was first introduced in [12]. It can be computed due to an Abelian version of the main result of [38]. This invariant is used in [41] to distinguish an arithmetic Zariski pair. It appears that this pair also has the property to possess non-isomorphic fundamental groups [11]. In the last section, two generalizations are given. First, a refinement of the $\mathcal{I}$-invariant introduced by Cadegan-Schlieper in his Ph.D. [18], then studied in [45]; second, a generalization to algebraic plane curves defined in [46], and used in [47, 15] to distinguish the Shimada's curves [76] and the k-Artal curves [5].

[9]The notations used here are the one of [18, 45].

2.1. The $\mathcal{I}$-invariant.

A *character* on an arrangement $\mathcal{A}$ is a morphism $\xi : \mathrm{H}_1(M(\mathcal{A});\mathbb{Z}) \to \mathbb{C}^*$. It is a *torsion character* if all the images are roots of unity. Due to Orlik and Solomon Theorem [66], $\mathrm{H}_1(M(\mathcal{A});\mathbb{Z})$ is the Abelian group generated by the homology class of the meridians m_ℓ, with the unique relation $\sum_{\ell\in\mathcal{A}} m_\ell = 0$. So, a character ξ can be described by associating to each line $\ell \in \mathcal{A}$ a non-zero complex number $\xi(m_\ell)$, such that $\prod_{\ell\in\mathcal{A}} \xi(m_\ell) = 1$. As a consequence, a character is a combinatorial object.

The *incidence graph* $\Gamma(\mathcal{A})$ associated to $\mathcal{A}$ is the bipartite graph given by:

- the first set of vertices is composed of the *point-vertices* v_P, for $\mathcal{A}_P \in \mathcal{C}(\mathcal{A})$,
- the second set of vertices is composed of the *line-vertices* v_ℓ, for $\ell \in \mathcal{A}$,
- the (oriented) edges of $\Gamma(\mathcal{A})$, denoted by $(P \to \ell)$, join v_P to v_ℓ if and only if $\ell \in \mathcal{A}_P$.

A *cycle* γ of $\Gamma(\mathcal{A})$ is an element of $\mathrm{H}_1(\Gamma(\mathcal{A});\mathbb{Z})$. So, it can be expressed as a chain of oriented edges of $\Gamma(\mathcal{A})$:

$$\gamma = \sum_{(P\to\ell)\in\Gamma(\mathcal{A})} a_{(P\to\ell)}(P \to \ell), \text{ with } a_{(P\to\ell)} \in \mathbb{Z},$$

and which verifies the boundary condition: $\sum_{(P\to\ell)\in\Gamma(\mathcal{A})} a_{(P\to\ell)}(v_\ell - v_P) = 0$. It is worth noticing that both $\Gamma(\mathcal{A})$ and γ are also combinatorial objects.

Definition 2.1. An *inner-cyclic triple* $(\mathcal{A}, \xi, \gamma)$ is formed by an arrangement $\mathcal{A}$, a character ξ on $\mathcal{A}$ and a cycle $\gamma \in \mathrm{H}_1(\Gamma(\mathcal{A});\mathbb{Z})$ such that:

(ICT1) for all $(P \to \ell) \in \Gamma(\mathcal{A})$, if $a_{(P\to\ell)} \neq 0$ then $\xi(m_{\ell'}) = 1$ for all $\ell' \in \mathcal{A}_P$,
(ICT2) for all $(P \to \ell) \in \Gamma(\mathcal{A})$ and $P' \in \mathrm{Sing}(\mathcal{A})$, if $a_{(P\to\ell)} \neq 0$ and $\ell \in \mathcal{A}_{P'}$ then $\prod_{\ell'\in\mathcal{A}_{P'}} \xi(m_{\ell'}) = 1$.

Two inner-cyclic triples $(\mathcal{A}_1, \xi_1, \gamma_2)$ and $(\mathcal{A}_2, \xi_2, \gamma_2)$ are *combinatorially equivalent* if $\mathcal{C}(\mathcal{A}_1) \sim \mathcal{C}(\mathcal{A}_2)$ and this equivalence sends ξ_1 on ξ_2 and γ_1 on γ_2,

The boundary manifold $B(\mathcal{A})$ of $\mathcal{A}$ is the boundary of a regular tubular neighborhood of $\mathcal{Z}(\mathcal{A})$. From Neumann [62] and Westlund [84], $B(\mathcal{A})$ is a graph manifold based on $\Gamma(\mathcal{A})$. So, it is determined by the combinatorics of $\mathcal{A}$. A *coherent embedding* of $\Gamma(\mathcal{A})$ in $B(\mathcal{A})$ is an embedding as described in [81, Section 9]. Basically, such an embedding sends the edge $(P \to \ell)$ in the boundary of a tubular neighborhood of ℓ. The embedding of $\Gamma(\mathcal{A})$ in $B(\mathcal{A})$ described in [38] is an explicit example of such a coherent embedding.

Let $j : \mathrm{H}_1(\Gamma(\mathcal{A});\mathbb{Z}) \to \mathrm{H}_1(B(\mathcal{A});\mathbb{Z})$ and $i : \mathrm{H}_1(B(\mathcal{A});\mathbb{Z}) \to \mathrm{H}_1(M(\mathcal{A});\mathbb{Z})$ be respectively the map induced by a coherent embedding $\Gamma(\mathcal{A}) \hookrightarrow B(\mathcal{A})$ and the inclusion $B(\mathcal{A}) \hookrightarrow M(\mathcal{A})$, on the first homology groups. We denote by $\Psi : \mathrm{H}_1(\Gamma(\mathcal{A});\mathbb{Z}) \to \mathrm{H}_1(M(\mathcal{A});\mathbb{Z})$ the composed map $i \circ j$.

Definition 2.2. Let $(\mathcal{A}, \xi, \gamma)$ be an inner-cyclic triple. The associated $\mathcal{I}$-invariant is given by:

$$\mathcal{I}(\mathcal{A}, \xi, \gamma) = \xi \circ \Psi(\gamma) \in \mathbb{C}^*.$$

It is crucial to note that $\mathcal{I}(\mathcal{A}, \xi, \gamma)$ is independent of the choice of the coherent embedding. Indeed, let $(\mathcal{A}, \xi, \gamma)$ be an inner-cyclic triple, if j_1 and j_2 are two different coherent embeddings then due to Conditions (ICT1) and (ICT2) one has $i \circ j_1(\gamma) - i \circ j_2(\gamma) \in \ker(\xi)$.

It follows from this construction that $\mathcal{I}(\mathcal{A}, \xi, \gamma)$ is an invariant of the ordered and oriented homeomorphism type of the complement $M(\mathcal{A})$. In section 2.3, an example of Zariski pair distinguished using this $\mathcal{I}$-invariant is exhibited.

Theorem 2.3 ([12, 45])**.** *If $(\mathcal{A}_1, \xi_1, \gamma_1)$ and $(\mathcal{A}_2, \xi_2, \gamma_2)$ are two combinatorially equivalent inner-cyclic triples such that any automorphism of $\Gamma(\mathcal{A}_i)$ respects its bipartite structure, and $M(\mathcal{A}_1)$ and $M(\mathcal{A}_2)$ have equivalent ordered and oriented topologies*[10]*, then one has:*

$$\mathcal{I}(\mathcal{A}_1, \xi_1, \gamma_1) = \mathcal{I}(\mathcal{A}_2, \xi_2, \gamma_2).$$

Remark 2.4. In [12], it is proven that the $\mathcal{I}$-invariant is an invariant of the (ordered and oriented) topology of the pair $(\mathbb{CP}^2, \mathcal{Z}(\mathcal{A}))$. The improvement to an invariant of the complement $M(\mathcal{A})$ is obtained in [45].

Remark 2.5. The $\mathcal{I}$-invariant has been generalized by Cadegan-Schlieper in his PhD thesis [18]. The main idea is to take elements in $\mathrm{H}^1(M(\mathcal{A}); G) \otimes_{\mathbb{Z}} \mathrm{H}_1(\Gamma(\mathcal{A}); \mathbb{Z})$ for an Abelian group G, rather than considering $(\xi, \gamma) \in \mathrm{H}^1(M(\mathcal{A}); \mathbb{C}^*) \times \mathrm{H}_1(\Gamma(\mathcal{A}); \mathbb{Z})$. This allows to reduce the Conditions (ICT1) and (ICT2). This generalization is studied in [45] and has been named the *loop-linking number*. More details are given in Section 2.4.1

Idea of the proof. Assume that there exists a homeomorphism $h : M(\mathcal{A}_1) \to M(\mathcal{A}_2)$. Then h induces a homeomorphism $h^B : B(\mathcal{A}_1) \to B(\mathcal{A}_2)$ which respects the graph structures of the boundary manifolds, see [82, 81]. This implies that we have the following commutative diagram.

$$\begin{tikzcd}
\mathrm{H}_1(\Gamma(\mathcal{A}_1)) \arrow[r, "j_1"] \arrow[d, "\mathrm{Id}"] & \mathrm{H}_1(B(\mathcal{A}_1)) \arrow[r, "i_1"] \arrow[d, "h^B_*"] & \mathrm{H}_1(M(\mathcal{A}_1)) \arrow[d, "h_*"] \\
\mathrm{H}_1(\Gamma(\mathcal{A}_2)) \arrow[r, "j_2"] & \mathrm{H}_1(B(\mathcal{A}_2)) \arrow[r, "i_2"] & \mathrm{H}_1(M(\mathcal{A}_2))
\end{tikzcd}$$

The result then follows from the independence of $\mathcal{I}(\mathcal{A}, \xi, \gamma)$ from the coherent embeddings j_1 and j_2. □

Since the lines of $\mathcal{A}$ are defined by complex 1-forms α_i, one can define $\overline{\mathcal{A}}$ the *complex conjugate arrangement* of $\mathcal{A}$, as the arrangement formed by the lines with complex conjugate equations. If $\mathcal{A}$ is a complexified real arrangement then $\overline{\mathcal{A}} = \mathcal{A}$. Remark that $\mathcal{A}$ and $\overline{\mathcal{A}}$ are combinatorially equivalent.

Proposition 2.6. *Let $(\mathcal{A}, \xi, \gamma)$ be an inner-cyclic triple. The triple $(\overline{\mathcal{A}}, \xi, \gamma)$ is inner-cyclic and combinatorially-equivalent to $(\mathcal{A}, \xi, \gamma)$. One has that*

$$\mathcal{I}(\mathcal{A}, \xi, \gamma)^{-1} = \mathcal{I}(\overline{\mathcal{A}}, \xi, \gamma).$$

In particular, if $\mathcal{A}$ is a complexified real arrangement then $\mathcal{I}(\mathcal{A}, \xi, \gamma) \in \{-1, 1\}$.

From this proposition, the following question naturally appears.

Question 2.7. *Does it exist Zariski pairs of complexified real arrangements distinguished by the $\mathcal{I}$-invariant?*

2.2. **Methods of computation.**

In [38], the inclusion map $i_* : \pi_1(B(\mathcal{A})) \to \pi_1(M(\mathcal{A}))$ is explicitly described. An Abelian version is used in this section to compute the $\mathcal{I}$-invariant. In the particular case of complexified real arrangements, an efficient diagrammatic method is given in Section 4.

[10] We refer to [45, Section 2.2] for the definition of the ordered and oriented topology of an arrangement complements.

2.2.1. *Via the braid monodromy.*

Let P_0 be a point of $M(\mathcal{A})$, and F_0 be a line passing through P_0 generic with $\mathcal{Z}(\mathcal{A})$, i.e. $\#F_0 \cap \mathcal{Z}(\mathcal{A}) = n$. Let $q : \mathbb{CP}^2 \setminus P_0 \to \mathbb{CP}^1$ be the natural projection defined by P_0. For each edge $(P \to \ell)$ of $\Gamma_{\mathcal{A}}$, we define the geometric braid $B_{(P\to\ell)}$ with $n - m(P) + 1$ strands as follows. Let R_P be a smooth path (without self-intersection) in $\mathbb{CP}^1$ from $q_0 := q(F_0 \setminus P_0)$ to $q(P)$ and such that $q(\mathrm{Sing}(\mathcal{A})) \cap R_P = q(P)$. This last condition implies that the $\mathbb{C}$-fiber over any point of R_P intersects $\mathcal{A}_{(P\to\ell)} = \{\ell\} \cup (\mathcal{A} \setminus \mathcal{A}_P)$ in exactly $n - m(P) + 1$ points. So, the geometric braid $B_{(P\to\ell)} \in \mathbb{B}_{n-m(P)+1}$ is defined as

$$B_{(P\to\ell)} = \mathcal{Z}(\mathcal{A}_{(P\to\ell)}) \cap q^{-1}(R_P) \subset R_P \times \mathbb{C}.$$

Remark 2.8. If a braid monodromy of $\mathcal{A}$ based in P_0 is given by $(b_1 t_1 b_1^{-1}, \cdots, b_m t_m b_m^{-1})$, where t_i is the local full-twist associated to the singular point $P_i \in \mathrm{Sing}(\mathcal{A})$ (see [60, 21] for details about braid monodromy), then $B_{P_i \to L}$ is the sub-braid of b_i obtained by removing the strands associated to the lines of $\mathcal{A}_{P_i} \setminus \ell$.

For a fixed system of coordinates, let $\Re : R_P \times \mathbb{C} \to R_P \times \mathbb{R}$ be the projection on the real part of the term $\mathbb{C}$ in $R_P \times \mathbb{C}$. Up to a slight perturbation, one can assume that $\Re(B_{(P\to\ell)})$ has only double points. So, the braid diagram of $B_{(P\to\ell)}$ associated to $\Re$ can be given as $\sigma_{j_1}^{\varepsilon_1} \cdots \sigma_{j_k}^{\varepsilon_k}$, with $\varepsilon_i \in \{-1, 1\}$, and $\sigma_1, \dots, \sigma_{n-m(P)}$ are the classical generators of the braid group $\mathbb{B}_{n-m(P)+1}$. Let $\mathrm{ulk}_\ell(B_{(P\to\ell)})$ be the *upper-linking* of ℓ with $B_{(P\to\ell)}$ defined by

$$\mathrm{ulk}_\ell(B_{(P\to\ell)}) = \sum_{i=1}^{k} \varepsilon_i . \delta_\ell(\sigma_i^{\varepsilon_i}),$$

where $\delta_\ell(\sigma_i^{\varepsilon_i})$ is the meridian of the strand over-crossing in $\sigma_i^{\varepsilon_i}$ if the under-crossing strand is associated to ℓ, otherwise it is 0.

Theorem 2.9. *Let $(\mathcal{A}, \xi, \gamma)$ be an inner-cyclic triple. One has the following expression for the $\mathcal{I}$-invariant:*

$$\mathcal{I}(\mathcal{A}, \xi, \gamma) = \prod_{(P\to\ell)\in\Gamma(\mathcal{A})} \xi\big(\mathrm{ulk}_\ell(B_{(P\to\ell)})\big)^{a_{(P\to\ell)}}.$$

Remark 2.10. A similar expression for the loop-linking number is given in [45, Theorem 3.2].

Idea of the proof. In [38], an explicit description of the map $i_* : \pi_1(B(\mathcal{A})) \to \pi_1(M(\mathcal{A}))$ is given. So, we consider an Abelian version of i_* to compute the value of $\Psi(\gamma)$. Fix a coherent embedding e as described in [38], and assume that γ is given by:

$$\gamma = \big((P_{i,k} \to \ell_i) - (P_{i,j} \to \ell_i)\big) + \big((P_{i,j} \to \ell_j) - (P_{j,k} \to \ell_j)\big) + \big((P_{j,k} \to \ell_k) - (P_{i,k} \to \ell_k)\big),$$

i.e. γ is a triangle in $\mathcal{Z}(\mathcal{A})$. To compute $\Psi(\gamma)$, we attach a 2-cell along $e(\gamma)$ and study its intersection with the lines of $\mathcal{A} \setminus \{\ell_i, \ell_j, \ell_k\}$. It appears that the term $\mathrm{ulk}_\ell(B_{(P\to\ell)})$ corresponds to the contribution of an edge $(P \to \ell)$ in γ.

Since triangular cycles generate $\mathrm{H}_1(\Gamma(\mathcal{A}))$, then one can use the previous description to compute the value of $\Psi(\gamma)$ for any $\gamma \in \mathrm{H}_1(\Gamma(\mathcal{A}))$, and one has

$$\Psi(\gamma) = \sum_{(P\to\ell)\in\Gamma(\mathcal{A})} a_{(P\to\ell)} . \,\mathrm{ulk}_\ell(B_{(P\to\ell)}).$$

The expression then follows from the definition of $\mathcal{I}(\mathcal{A}, \xi, \gamma)$. $\square$

2.2.2. *Via the braided wiring diagram.*

Roughly speaking, the *braided wiring diagram*, or shortly the *wiring diagram*, is the trace of the arrangement $\mathcal{A}$ in the fibers over a smooth path $\rho : [0,1] \to \mathbb{CP}^1$ starting from q_0 and passing through all the points of $q(\mathrm{Sing}(\mathcal{A}))$, see [14, 21] for details braided wiring diagrams. It is a singular braid, whose singular points correspond to the singular points of $\mathcal{A}$.

We order the points $\{P_1, \ldots, P_m\}$ of $\mathrm{Sing}(\mathcal{A})$ according to the order of their image in ρ, and we re-parametrized ρ such that $\rho(i/m) = P_i$, for $i \in \{0, \cdots, m\}$. A wiring diagram $\mathcal{W}(\mathcal{A})$ of $\mathcal{A}$ can be given as an ordered m-tuple of pairs formed by a braid $b_i \in \mathbb{B}_n$ and a singular point $P_i \in \mathrm{Sing}(\mathcal{A})$:

$$\mathcal{W}(\mathcal{A}) = \big[\,[b_1, P_1], \cdots, [b_m, P_m]\,\big], \tag{WD}$$

where $b_i = \mathcal{A} \cap q^{-1}\big(\widehat{P_{i-1}P_i}\big) \in \mathbb{B}_n$, with $\widehat{P_{i-1}P_i} = \rho\big((\frac{i-1}{m} + \varepsilon, \frac{i}{m} - \varepsilon)\big)$ for ε small enough.

For a fixed $i \in \{1, \cdots, m\}$ and $\ell \in \mathcal{A}_{P_i}$, the braid $B_{(P_i \to \ell)}$ can be obtained from $\mathcal{W}(\mathcal{A})$ as follows. Consider that the path R_{P_i} defined in the previous section is a slight deformation of $\rho\big((0, \frac{i}{m})\big)$ which avoids the points $q(P_1), \ldots, q(P_{i-1})$ turning around them counter-clockwise. In such a situation, we define:

$$\overline{B}_{P_i \to \ell} = b_1 \cdot T_1 \cdot b_2 \cdot \cdots \cdot b_{i-1} \cdot T_{i-1} \cdot b_i,$$

where T_j is the local positive half-twist of the strands which correspond to the lines of $\mathcal{A}_{P_j}$, see [12, Section 4] for an explicit example. The braid $B_{(P_i \to \ell)}$ is obtained from $\overline{B}_{(P_i \to \ell)}$ by removing the strands which correspond to the lines $\mathcal{A}_{P_i}$, except L.

The computation is completed using Theorem 2.9. Due to Remark 2.10, it is also possible to compute the loop-linking number using this method.

2.3. Application: detection of an arithmetic Zariski pair.

Consider the combinatorics $\mathcal{K}_{11} = (\mathcal{L}, \mathcal{P})$ defined by $\mathcal{L} = \{\ell_1, \cdots, \ell_{11}\}$ and

$$\mathcal{P} = \left\{ \begin{array}{c} \{\ell_1,\ell_2\}, \{\ell_1,\ell_3\}, \{\ell_1,\ell_4,\ell_5,\ell_6\}, \{\ell_1,\ell_7,\ell_{11}\}, \{\ell_1,\ell_8,\ell_9,\ell_{10}\}, \{\ell_2,\ell_3\}, \{\ell_2,\ell_4,\ell_{10},\ell_{11}\} \\ \{\ell_2,\ell_5,\ell_9\}, \{\ell_2,\ell_6,\ell_7,\ell_8\}, \{\ell_3,\ell_4,\ell_9\}, \{\ell_3,\ell_5,\ell_8\}, \{\ell_3,\ell_6,\ell_{11}\}, \{\ell_3,\ell_7,\ell_{10}\}, \{\ell_4,\ell_7\} \\ \{\ell_4,\ell_8\}, \{\ell_5,\ell_7\}, \{\ell_5,\ell_{10}\}, \{\ell_5,\ell_{11}\}, \{\ell_6,\ell_9\}, \{\ell_6,\ell_{10}\}, \{\ell_7,\ell_9\}, \{\ell_8,\ell_{11}\}, \{\ell_9,\ell_{11}\} \end{array} \right\}$$

Proposition 2.11. *The automorphism group of $\mathcal{K}_{11}$ is cyclic of order 4, and it is generated by:*

$$\sigma = (\ell_1, \ell_2)(\ell_4, \ell_6, \ell_8, \ell_{10})(\ell_5, \ell_7, \ell_9, \ell_{11}).$$

The arrangements defined by the following equations admit $\mathcal{K}_{11}$ as combinatorics:

$$\begin{array}{ll} \ell_1 : z = 0, & \ell_2 : x + y - z = 0, \\ \ell_3 : \alpha x + y = 0, & \ell_4 : \alpha x + z = 0, \\ \ell_5 : x = 0, & \ell_6 : x - z = 0, \\ \ell_7 : -x + \alpha^2 y + z = 0, & \ell_8 : y = 0, \\ \ell_9 : y - z = 0, & \ell_{10} : \alpha y - (\alpha+1)z = 0, \\ \ell_{11} : -x + \alpha^2 y + (\alpha^3+1)z = 0. & \end{array}$$

where α is a root of the 5th cyclotomic polynomial $\Phi_5 = X^4 + X^3 + X^2 + X + 1$. We denoted these arrangements by $\mathcal{M}^{\pm}$ and $\mathcal{N}^{\pm}$ in such a way that $\mathcal{M}^+$ and $\mathcal{M}^-$ (resp. $\mathcal{N}^+$ and $\mathcal{N}^-$) are complex conjugates, and $\mathcal{M}^+$ (resp. $\mathcal{N}^+$) corresponds to the root $\alpha \approx -0.81 + 0.59i$ (resp. $\alpha \approx 0.31 + 0.95i$). Braided wiring diagrams of $\mathcal{M}^+$ and $\mathcal{N}^+$ are pictured in Figures (1) and (2) respectively.

Remark 2.12. There exists a projective transformation $T \in \mathrm{PGL}_3(\mathbb{C})$ which permutes cyclically $\mathcal{M}^+$, $\mathcal{N}^+$, $\mathcal{M}^-$ and $\mathcal{N}^-$. This transformation is a geometric realization of the generator σ of the automorphism group of $\mathcal{K}_{11}$. This implies that $\mathcal{M}^{\pm}$ and $\mathcal{N}^{\pm}$ have all the same topology.

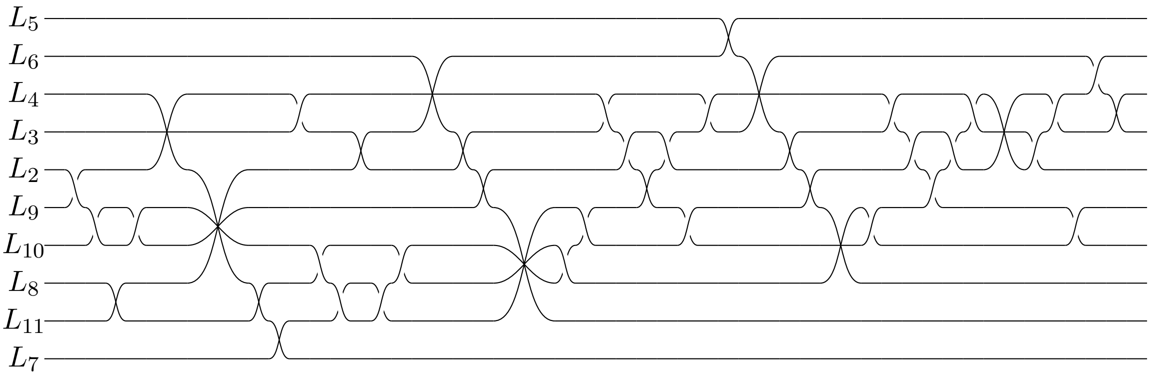


FIGURE 1. Wiring diagram of $\mathcal{M}^+$ with L_1 as the line at infinity.

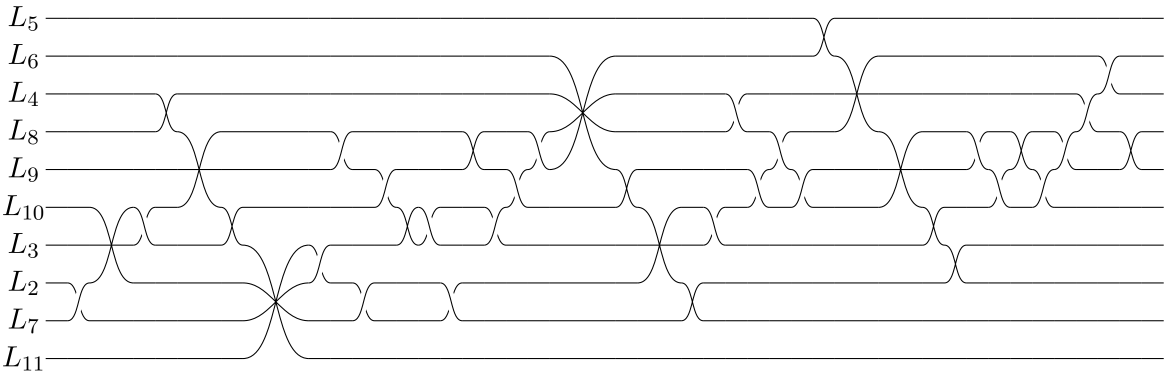


FIGURE 2. Wiring diagram of $\mathcal{N}^+$ with L_1 as the line at infinity.

Consider the following cycle $\gamma \in \mathrm{H}_1(\Gamma(\mathcal{K}))$:

$$\gamma = \big((P_{1,2} \to \ell_1) - (P_{1,3} \to \ell_1)\big) + \big((P_{2,3} \to \ell_2) - (P_{1,2} \to \ell_2)\big) + \big((P_{1,3} \to \ell_3) - (P_{2,3} \to \ell_3)\big),$$

and the character ξ defined by:

$$\xi : (\ell_1, \ldots, \ell_{11}) \longmapsto (1, 1, 1, \zeta, \zeta, \zeta^3, \zeta^3, \zeta^4, \zeta^4, \zeta^2, \zeta^2),$$

where ζ is a 5th root of unity.

Proposition 2.13. *The triples $(\mathcal{M}^{\pm}, \xi, \gamma)$ and $(\mathcal{N}^{\pm}, \xi, \gamma)$ are inner-cyclic triples.*

Theorem 2.14. *There is no homeomorphism preserving both orientation and order between any two pairs among $M(\mathcal{M}^+)$, $M(\mathcal{M}^-)$, $M(\mathcal{N}^+)$ and $M(\mathcal{N}^-)$.*

Proof. Let $\Psi_{\mathcal{A}}$ be the map from $\mathrm{H}_1(\Gamma(\mathcal{A}))$ to $\mathrm{H}(M(\mathcal{A}))$, for $\mathcal{A} = \mathcal{M}^+$ or $\mathcal{A} = \mathcal{N}^+$. Following the method given in Section 2.2.2 applied on the braided wiring diagrams given in Figures (1) and (2), one has that:

$$\Psi_{\mathcal{M}^+}(\gamma) = m_7 + m_8 + m_{10} \quad \text{and} \quad \Psi_{\mathcal{N}^+}(\gamma) = -m_7.$$

Then, we apply the character ξ and using Proposition 2.6, we obtain that:

$$\mathcal{I}(\mathcal{M}^+, \xi, \gamma) = \zeta^4, \quad \mathcal{I}(\mathcal{M}^-, \xi, \gamma) = \zeta, \quad \mathcal{I}(\mathcal{N}^+, \xi, \gamma) = \zeta^2, \quad \mathcal{I}(\mathcal{N}^-, \xi, \gamma) = \zeta^3.$$

We conclude using Theorem 2.3. □

Corollary 2.15. *There is no order-preserving homeomorphism between the complements $M(\mathcal{M}^{\pm})$ and $M(\mathcal{N}^{\pm})$.*

To remove the ordered condition in the previous lemma, we can apply a strategy similar to the one of [7]. That is to say, we add a line ℓ_{12} to the combinatorics $\mathcal{K}$ to obtain a combinatorics with a trivial automorphism group. Note that the choice of this line ℓ_{12} is not unique, we make an arbitrary one in the following paragraph.

By Proposition 2.11, we know that any non-trivial automorphism of the combinatorics $\mathcal{K}$ permutes the four points of multiplicity 4. So, we add a line ℓ_{12} which passes though $\{\ell_1, \ell_4, \ell_5, \ell_6\}$. To completely fix this line ℓ_{12}, we can also assume that it passes though $\{\ell_9, \ell_{11}\}$. We denote by $\mathfrak{M}^{\pm}$ and $\mathfrak{N}^{\pm}$, the arrangement $\mathcal{M}^{\pm}$ and $\mathcal{N}^{\pm}$ with such an additional line ℓ_{12}.

The arrangements $\mathfrak{M}^{\pm}$ and $\mathfrak{N}^{\pm}$ are combinatorially equivalent. Indeed, if the addition of this twelfth line creates other multiple points, then they will be present in all $\mathfrak{M}^{\pm}$ and $\mathfrak{N}^{\pm}$. This comes from the fact that the arrangements $\mathcal{M}^{\pm}$ and $\mathcal{N}^{\pm}$ are Galois conjugate. We deduce the following corollary, where the arrangements $\mathfrak{M}^{\pm}$ and $\mathfrak{N}^{\pm}$ are considered as non-order arrangements.

Corollary 2.16. *There is no homeomorphism between $M(\mathfrak{M}^{\pm})$ and $M(\mathfrak{N}^{\pm})$.*

A Zariski pair is said to be *arithmetic* if the equations of the arrangements are Galois-conjugated in a number field, so the arrangements $\mathfrak{M}^{\pm}$ and $\mathfrak{N}^{\pm}$ form arithmetic Zariski pairs. The topology of such pairs cannot be distinguished by algebraic arguments. In particular, their fundamental groups have the same profinite completions. In [11], the fundamental groups of the previous arrangements have been studied.

Theorem 2.17. *The fundamental groups $\pi_1(M(\mathfrak{M})^{\pm})$ and $\pi_1(M(\mathfrak{N})^{\pm})$ are not isomorphic.*

Idea of the proof. The first step is to prove the homological rigidity of these arrangements. It is a combinatorial property introduced by Marco in [57] which implies that any isomorphism between the fundamental groups induces the identity at the homological level. The second step is to apply the Alexander invariant isomorphism test of level 2 developed by Artal, Carmona, Cogolludo and Marco in [8] to distinguish the Rybnikov arrangements. □

This provides the first example of arithmetic Zariski pair with non-isomorphic fundamental groups, even in the larger context of algebraic plane curves. Other such examples are given in [45]. Their fundamental groups are distinguished using the same method.

2.4. Generalizations of the $\mathcal{I}$-invariant.

The $\mathcal{I}$-invariant has been generalized in two directions. The first one has been mentioned many times in the previous section, it is the *loop-linking number* introduced by Cadegan-Schlieper in his Ph.D. thesis [18]. It is studied in [45] and allows the distinguishing of several new Zariski pairs, but also the Rybnikov one, and so to solve a weak version of Falk and Randell Problem 1.2 in [36]. The second one is a generalization to algebraic plane curves, and it is simply named the *linking invariant* [46]. It has been studied in [47, 15]. In this section, we give a short overview of these two generalizations.

2.4.1. *Loop-linking numbers.*

Let G be an Abelian group. We consider the tensor space $\mathrm{H}^1(M(\mathcal{A}); G) \otimes_{\mathbb{Z}} \mathrm{H}_1(\Gamma(\mathcal{A}), \mathbb{Z})$. From Orlik-Solomon [66], this space is determined by the combinatorics. The *tensor linking group* of $\mathcal{A}$, denoted by $\mathrm{TLG}(\mathcal{A}, G)$, is the subgroup of $\mathrm{H}^1(M(\mathcal{A}); G) \otimes_{\mathbb{Z}} \mathrm{H}_1(\Gamma(\mathcal{A}), \mathbb{Z})$ formed by the elements which verify:

(TLG1) for all $(P \to \ell)$, and all $\ell' \in \mathcal{A}$ containing P, we have $\lambda_{(P\to\ell)}(m_{\ell'}) = 0_G$,
(TLG2) for all $(P \to \ell)$, and all $P' \in \mathrm{Sing}(\mathcal{A})$ contained in ℓ, we have $\sum_{\ell' \ni P'} \lambda_{(P\to\ell)}(m_{\ell'}) = 0_G$.

Using the notations introduced in Section 2.3, we define the map $\Psi : \mathrm{H}^1(M(\mathcal{A}); G) \otimes_{\mathbb{Z}} \mathrm{H}_1(\Gamma(\mathcal{A}), \mathbb{Z}) \to \mathrm{H}^1(M(\mathcal{A}); G) \otimes_{\mathbb{Z}} \mathrm{H}_1(M(\mathcal{A}); \mathbb{Z})$ by $\Psi = \mathrm{Id}_{\mathrm{H}^1(M(A);G)} \otimes (i \circ j)$. The natural pairing of $\mathrm{H}^1(M(\mathcal{A}); G) \otimes_{\mathbb{Z}} \mathrm{H}_1(M(\mathcal{A}); \mathbb{Z})$ is denoted by π.

Definition 2.18. The *loop-linking number* of $\mathcal{A}$ associated to $\Lambda \in \mathrm{TLG}(\mathcal{A}, G)$ is

$$\mathcal{L}(\mathcal{A}, \Lambda) = \pi \circ \Psi(\Lambda) \in G.$$

It is well-defined since the difference between two coherent embeddings vanishes when we take the pairing π due to Conditions (TLG1) and (TLG2). For more details about this definition, we refer to [18, Section 3.3.1]. Cadegan-Schlieper proves that the loop-linking number is an invariant of the ordered and oriented topology, his result is improved in [45] where it is proven that it is an invariant of the ordered and oriented homeomorphism type of the complement.

Theorem 2.19 (Proposition 21 in [18] and Theorem 2.6 [45])**.** *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two ordered line arrangements. If there exists a homeomorphism h from $M(\mathcal{A}_1)$ to $M(\mathcal{A}_2)$, which preserves the orientation and the orders, then for any $\Lambda \in \mathrm{TLG}(\mathcal{A}_1, G)$,*

$$\mathcal{L}(\mathcal{A}_1, \Lambda) = \mathcal{L}(\mathcal{A}_2, h_*(\Lambda)),$$

where $h_* : \mathrm{TLG}(\mathcal{A}_1, G) \to \mathrm{TLG}(\mathcal{A}_2, G)$ is the isomorphism induced by h.

2.4.2. *Linking invariant of algebraic plane curves.*

Let $\mathcal{C} \cup \mathcal{D}$ be a reducible algebraic curve, decomposed into two nonempty subcurves $\mathcal{C}$ and $\mathcal{D}$. Consider the inclusion maps $i : \mathcal{C} \setminus \mathcal{D} \hookrightarrow \mathcal{C}$ and $j : \mathcal{C} \setminus \mathcal{D} \hookrightarrow \mathbb{CP}^2 \setminus \mathcal{D}$, and denote respectively by i_* and j_* the induced map on the first homology groups. Note that

$$\ker(i_*) \simeq \mathrm{H}_1(\partial \bigcup_{C \in \mathrm{Irr}(\mathcal{C})} C \setminus \mathcal{D}) \simeq \bigoplus_{C \in \mathrm{Irr}(\mathcal{C})} \mathrm{H}_1\left(\partial(C \setminus \mathcal{D})\right) \subset \mathrm{H}_1(\mathcal{C} \setminus \mathcal{D}).$$

Definition 2.20. The *indeterminacy subgroup* with respect to $\mathcal{C}$, denoted by $\mathcal{J}_{\mathcal{C}}$, is the subgroup of $\mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{D})$ defined as the image of $\bigoplus_{C \in \mathrm{Irr}(\mathcal{C})} \mathrm{H}_1\left(\partial(C \setminus \mathcal{D})\right)$ by j_*.

Let $\Gamma(\mathcal{C} \cup \mathcal{D})$ be the incidence graph of the irreducible components of $\mathcal{C} \cup \mathcal{D}$ defined similarly than for line arrangements in Section 2.1. This graph can naturally be embedded in $\mathcal{C} \cup \mathcal{D}$. A cycle $\gamma \in \Gamma(\mathcal{C})$ is said to avoid $\mathcal{D}$ if its image by the inclusion $\Gamma(\mathcal{C}) \hookrightarrow \Gamma(\mathcal{C} \cup \mathcal{D})$ avoid any vertex (as well point-vertex as component-vertex) associated to $\mathcal{D}$. We denote by $[\gamma]$ the class in $\mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{D})/\mathcal{J}_{\mathcal{C}}$ of the image of a cycle $\gamma \in \Gamma(\mathcal{C})$ in $\mathcal{C} \setminus \mathcal{D} \subset \mathbb{CP}^2 \setminus \mathcal{D}$. We also denote by $\mathcal{I}_{\mathcal{C}}$ the image of $\bigoplus_{C \in \mathrm{Irr}(\mathcal{C})} \mathrm{H}_1(C \setminus \mathcal{D})$ by j_*, composed with the projection map $\mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{D}) \to \mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{D})/\mathcal{J}_{\mathcal{C}}$.

Definition 2.21. The *oriented linking of $\mathcal{C}$ with $\mathcal{D}$ along γ*, denoted by $\mathrm{lk}_\gamma(\mathcal{C}, \mathcal{D})$, is the coset of $\mathcal{I}_{\mathcal{C}}$ in $\mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{D})/\mathcal{J}_{\mathcal{C}}$ with respect to $[\gamma]$. In other words,

$$\mathrm{lk}_\gamma(\mathcal{C}, \mathcal{D}) = [\gamma]\mathcal{I}_{\mathcal{C}} \ \subset \ \mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{D})/\mathcal{J}_{\mathcal{C}}.$$

Note that the above formula is well-defined, *i.e.* $\mathrm{lk}_\gamma(\mathcal{C}, \mathcal{D})$ does not depend on the choice of the embedding of $\Gamma(\mathcal{C} \cup \mathcal{D})$ in $\mathcal{C} \cup \mathcal{D}$. Furthermore, we have the following description of $\mathcal{J}_{\mathcal{C}}$.

Proposition 2.22. *The indeterminacy subgroup $\mathcal{J}_{\mathcal{C}}$ is spanned by the elements of the form:*

$$\sum_{d \in \mathcal{D}(P)} I_P(b, d).m_{\beta_P(d)}, \quad \textit{for all } P \in \mathcal{C} \cap \mathcal{D} \textit{ and all } b \in \mathcal{C}(P),$$

where $I_P(b, d)$ denotes the intersection multiplicity of the local branches b and d at P, and $m_{\beta_P(d)}$ is given by a meridian of the irreducible component $\beta_P(d)$ of $\mathcal{D}$ containing d.

As a consequence of Proposition 2.22, the group $\mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{D})/\mathcal{J}_{\mathcal{C}}$ is determined by the combinatorics of the curve $\mathcal{C} \cup \mathcal{D}$. So we can use the linking invariant to compare the topology of curves with the same combinatorics. Indeed, we have the following theorem, which implies that the linking invariant is an invariant of the oriented topology of $(\mathbb{CP}^2, \mathcal{C} \cup \mathcal{D})$.

Theorem 2.23. *Let ϕ be an orientation-preserving homeomorphism between two algebraic curves $\mathcal{C}_1 \cup \mathcal{D}_1$ and $\mathcal{C}_2 \cup \mathcal{D}_2$. It induces an isomorphism ϕ_* between $\mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{D}_1)/\mathcal{J}_{\mathcal{C}_1}$ and $\mathrm{H}_1(\mathbb{CP}^2 \setminus \mathcal{D}_2)/\mathcal{J}_{\mathcal{C}_2}$, and for any cycle $\gamma_1 \in \Gamma_{\mathcal{C}_1}$ avoiding $\mathcal{D}_1$, we have*

$$\phi_*(\mathrm{lk}_{\gamma_1}(\mathcal{C}_1, \mathcal{D}_1)) = \mathrm{lk}_{\phi_\Gamma(\gamma_1)}(\mathcal{C}_2, \mathcal{D}_2).$$

This linking invariant has been used to distinguish several Zariski pairs of algebraic plane curves. In [46], it distinguishes the 3-Artal curves [5]. These curves are formed by a smooth cubic and 3 inflectional tangent lines, but also the quartic with 3 bitangents. Then, in [15], it classifies the topologies of all the k-Artal curves. Finally, in [47], an equivalence between this linking invariant and the splitting number introduced by Shirane in [77] is obtained. It allows us to prove that this linking invariant for curves is not determined by the fundamental group of the complements. Indeed, it distinguishes the Shimada's curves, which are known to be π_1-equivalent [76]. This property still holds for the $\mathcal{I}$-invariant as shown in Corollary 5.17.

3. Gluing of arrangements & Multiplicativity theorem

Inspired by the idea of Rybnikov in [71, 72], where he constructs the first known example of a Zariski pair by gluing together two MacLane arrangements, we present in this section a multiplicativity theorem for the $\mathcal{I}$-invariant under the gluing of two arrangements along a triangle. This result first appears in [42]. It is then extended to the loop-linking invariant in [45]. In this context, it allows us to distinguish the homeomorphism type of the complements of Rybnikov arrangements. This provides a solution to a lighter version of Falk and Randell Problem 1.2 in [36]:

Problem. *Find a general invariant of arrangement complements that distinguishes the two Rybnikov arrangements, and generalize his construction.*

3.1. Definitions & statement.

An inner-cyclic triple $(\mathcal{A}, \xi, \gamma)$ is triangular if γ contains exactly three line-vertices. Throughout all this section, when we consider a triangular we assume that the line vertices of γ are, v_{ℓ_1}, v_{ℓ_2} and v_{ℓ_3}. Up to a relabelling, this is always possible.

Let $\mathcal{A}_1 = \{\ell_1^1, \dots, \ell_n^1\}$ and $\mathcal{A}_2 = \{\ell_1^2, \dots, \ell_m^2\}$ be two ordered arrangements (the order is given by the indices). A *gluing* of $\mathcal{A}_1$ with $\mathcal{A}_2$ is a projective transformation ϕ such that

- $\phi(\ell_1^2) = \ell_1^1$, $\phi(\ell_2^2) = \ell_2^1$ and $\phi(\ell_3^2) = \ell_3^1$,
- for all $i \in \{4, \dots, m\}$, $\phi(\ell_i^2) \notin \mathcal{A}_1$.

The *glued arrangement* of $\mathcal{A}_1$ with $\mathcal{A}_2$ associated with ϕ is the ordered arrangement[11]:

$$\mathcal{A}_1 \triangle_\phi \mathcal{A}_2 = \{\ell_1^1, \dots, \ell_n^1, \phi(\ell_4^2), \dots, \phi(\ell_m^2)\},$$

ordered as in the set above. A gluing ϕ is *generic* if for any small perturbation ϕ' of ϕ which is also a gluing, the arrangements $\mathcal{A}_1 \triangle_\phi \mathcal{A}_2$ and $\mathcal{A}_1 \triangle_{\phi'} \mathcal{A}_2$ are combinatorially equivalent. In such case, $\mathcal{A}_1 \triangle_\phi \mathcal{A}_2$ is called a *generic glued arrangement*.

[11]Note that the gluing operation is not commutative for order arrangements.

The lines of $\mathcal{A}_1 \triangle_\phi \mathcal{A}_2$ are denoted by $d_1, \dots, d_k$, with $k = n + m - 3$, so that the order of $\mathcal{A}_1 \triangle_\phi \mathcal{A}_2$ coincides with the indices. Their associated meridians in $M(\mathcal{A}_1 \triangle_\phi \mathcal{A}_2)$ are denoted by $\mathfrak{m}_1, \dots, \mathfrak{m}_k$ respectively. The meridian of ℓ_i^j in $\mathrm{H}_1(M(\mathcal{A}_j))$ is denoted by m_i^j. Let $\xi_1 \in \mathrm{H}^1(M(\mathcal{A}_1))$ and $\xi_2 \in \mathrm{H}^1(M(\mathcal{A}_2))$ be two characters, and let ϕ be a gluing of $\mathcal{A}_1$ with $\mathcal{A}_2$. We define the *glued character* $\xi_1 \triangle \xi_2$ by:

$$\xi_1 \triangle \xi_2 : \left\{ \begin{array}{ccll} \mathrm{H}_1(M(\mathcal{A}_1 \triangle_\phi \mathcal{A}_2)) & \longrightarrow & \mathbb{C}^* & \\ \mathfrak{m}_i & \longmapsto & 1 & , \text{for } i \in \{1,2,3\} \\ \mathfrak{m}_i & \longmapsto & \xi_1(m_i^1) & , \text{for } i \in \{4,\dots,n\} \\ \mathfrak{m}_i & \longmapsto & \xi_2(m_{i-n+3}^2) & , \text{for } i \in \{n+1,\dots,k\} \end{array} \right.$$

Note that by Condition (ICT1) in Definition 2.1, one has $\xi_1(m_i^1) = 1$ and $\xi_2(m_i^2) = 1$, for $i \in \{1,2,3\}$. So, for these three indices, one has $\xi_1 \triangle \xi_2(\mathfrak{m}_i) = \xi_1(m_i^1) \cdot \xi_2(m_i^2)$. When there is no ambiguity on the arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ (resp. on ξ_1 and ξ_2) the glued arrangement $\mathcal{A}_1 \triangle_\phi \mathcal{A}_2$ (resp. the glued character $\xi_1 \triangle \xi_2$) will be denoted by $\mathfrak{A}_\phi$ (resp. $\mathfrak{X}_\phi$).

Theorem 3.1. *Let $(\mathcal{A}_1, \xi_1, \gamma_1)$ and $(\mathcal{A}_2, \xi_2, \gamma_2)$ be two triangular inner-cyclic triples, and ϕ be a gluing of $\mathcal{A}_1$ with $\mathcal{A}_2$. The triple $(\mathfrak{A}_\phi, \mathfrak{X}, \mu)$ is also a triangular inner-cyclic triple (where μ is the cycle supported by the lines d_1, d_2 and d_3). Furthermore, one has:*

$$\mathcal{I}(\mathfrak{A}_\phi, \mathfrak{X}_\phi, \mu) = \mathcal{I}(\mathcal{A}_1, \xi_1, \gamma_1) \cdot \mathcal{I}(\mathcal{A}_2, \xi_2, \gamma_2).$$

Note that the gluing ϕ is not necessarily a generic gluing.

Idea of the proof. The set of conditions for $(\mathfrak{A}_\phi, \mathfrak{X}, \mu)$ to be a triangular inner-cyclic triple, is the union of the sets of conditions for $(\mathcal{A}_1, \xi_1, \gamma_1)$ and $(\mathcal{A}_2, \xi_2, \gamma_2)$ to be triangular inner-cyclic triples.

The multiplicativity relation comes from a similar fact, but with the expression of the $\mathcal{I}$-invariant given in Theorem 2.9. Indeed, the linking $\mathrm{ulk}_\ell(B_{(P \to \ell)})$ can be divided into two part the linking with the lines of $d_4, \dots, d_n$ and with the lines $d_{n+1}, \dots, d_k$. □

3.2. **Extended Rybnikov arrangements.**

We consider the extended MacLane arrangements denoted by $\mathcal{ML}_\pm^e$ introduced in [12]. They are formed by the MacLane arrangements $\mathcal{ML}_\pm$ with an additional line passing through two points of multiplicity 3 and one point of multiplicity 2. The combinatorics of the extended MacLane arrangements is:

$$\mathcal{C}(\mathcal{ML}_\pm^e) = \left\{ \begin{array}{c} \{\ell_1,\ell_2\}, \{\ell_1,\ell_3\}, \{\ell_1,\ell_4,\ell_5,\ell_6\}, \{\ell_1,\ell_7,\ell_8,\ell_9\} \\ \{\ell_2,\ell_3\}, \{\ell_2,\ell_4,\ell_9\}, \{\ell_2,\ell_5,\ell_8\}, \{\ell_2,\ell_6,\ell_7\}, \{\ell_3,\ell_4,\ell_7\} \\ \{\ell_3,\ell_5,\ell_9\}, \{\ell_3,\ell_6,\ell_8\}, \{\ell_4,\ell_8\}, \{\ell_5,\ell_7\}, \{\ell_6,\ell_9\} \end{array} \right\}.$$

The two realizations $\mathcal{ML}_+^e$ and $\mathcal{ML}_-^e$ are given by the equations:

$$\ell_1 : z = 0, \quad \ell_2 : x - a^2 y = 0, \quad \ell_3 : x - ay = 0, \quad \ell_4 : y - a^2 z = 0, \quad \ell_5 : y - z = 0,$$
$$\ell_6 : y - az = 0, \quad \ell_7 : x - z = 0, \quad \ell_8 : x - a^2 z = 0, \quad \ell_9 : x - az = 0,$$

where a is a root of the 3rd cyclotomic polynomial $X^2 + X + 1$, i.e. a is a primitive third root of unity.

Consider γ the cycle of $\mathrm{H}_1(\Gamma(\mathcal{C}(\mathcal{ML}_\pm^e)))$ with line-vertices v_{ℓ_1}, v_{ℓ_2} and v_{ℓ_3}. In addition, let $\xi \in \mathrm{H}^1(M(\mathcal{ML}_\pm^e))$ defined by:

$$\xi : (\ell_1, \dots, \ell_9) \longmapsto (1, 1, 1, \zeta, \zeta, \zeta, \zeta^2, \zeta^2, \zeta^2),$$

where $\zeta \in \mathbb{C}^*$ is a primitive third root of unity.

Proposition 3.2. *The triples* $(\mathcal{ML}^e_\pm, \xi, \gamma)$ *are triangular inner-cyclic triples. Furthermore, one has:*

$$\mathcal{I}(\mathcal{ML}^e_+, \xi, \gamma) = \zeta \quad \textit{and} \quad \mathcal{I}(\mathcal{ML}^e_-, \xi, \gamma) = \zeta^2.$$

This proposition implies that there is no order and oriented preserving homeomorphism between the complements of the extended MacLane arrangements. Nevertheless, the complex conjugation implies a non-orientation preserving homeomorphism.

In his paper [71, 72], Rybnikov introduces four arrangements constructed by gluing together two MacLane arrangements. Let us define, similarly, the extended Rybnikov arrangements[12] $\mathcal{R}^e_{\pm,\pm}$ and $\mathcal{R}^e_{\pm,\mp}$. Let ϕ^+ (resp. ϕ^-) be a generic gluing of $\mathcal{ML}^e_+$ with $\mathcal{ML}^e_+$ (resp. with $\mathcal{ML}^e_-$). One set:

$$\mathcal{R}^e_{+,\pm} = \mathcal{ML}^e_+ \triangle_{\phi^\pm} \mathcal{ML}^e_\pm.$$

Their complex conjugates are denoted by $\mathcal{R}^e_{-,\mp}$.

Theorem 3.3. *The four triples* $(\mathcal{R}^e_{\pm,\pm}, \xi\triangle\xi, \gamma)$ *and* $(\mathcal{R}^e_{\pm,\mp}, \xi\triangle\xi, \gamma)$ *are triangular inner-cyclic. Furthermore, one has:*

$$\mathcal{I}(\mathcal{R}^e_{\pm,\pm}, \xi\triangle\xi, \gamma) = \zeta^{\pm 1} \quad \textit{and} \quad \mathcal{I}(\mathcal{R}^e_{\pm,\mp}, \xi\triangle\xi, \gamma) = 1.$$

Corollary 3.4. *There is no order-preserving homeomorphism between* $M(\mathcal{R}^e_{\pm,\pm})$ *and* $M(\mathcal{R}^e_{\pm,\mp})$.

Using a strategy similar to Section 2.3, it is possible to add two lines to the extended Rybnikov arrangements and obtain a combinatorics with a trivial automorphism group. As non-ordered arrangements, there will be no homeomorphism between their complements.

One can generalize the previous construction as follows. Let $\mathcal{A}$ be an arrangement, and $\overline{\mathcal{A}}$ be its complex conjugate. We denote by $\mathfrak{A}_+$ (resp. $\mathfrak{A}_-$) the generic glued arrangement of $\mathcal{A}$ with itself (resp. $\overline{\mathcal{A}}$).

Theorem 3.5. *If* $(\mathcal{A}, \xi, \gamma)$ *is a triangular inner-cyclic arrangement such that* $\mathcal{I}(\mathcal{A}, \xi, \gamma)$ *is not real, then there is no order-preserving homeomorphism between* $M(\mathfrak{A}_+)$ *and* $M(\mathfrak{A}_-)$.

If, in addition, the automorphism group of $\mathcal{C}(\mathcal{A})$ *is trivial, then, as non-ordered arrangements, there is no homeomorphism between the complements of* $\mathfrak{A}_+$ *and* $\mathfrak{A}_-$.

As announced in the introduction of this section, the previous Theorem provides a solution to Falk and Randell Problem 1.2 in [36].

4. Configurations of points & topology of their dual arrangements

The results presented in this section provide a positive answer to Question 2.7. To reach this goal, we developed a diagrammatic version of the $\mathcal{I}$-invariant in the particular case of complexified real arrangements [48]. The Zariski pairs obtained here are the first ones with rational coefficients, but also whose topologies are distinguished without computer assistance. Furthermore, the connected components of their moduli spaces can be geometrically characterized. To our knowledge, that was also the first time that such a phenomenon was observed.

[12] The gluing performed by Rybnikov is slightly different from ours. Indeed, he glued the two MacLane arrangements along three concurrent lines, while we are gluing the extended MacLane arrangements along three lines in generic position. In particular, this implies that Rybnikov arrangements are not subarrangements of the extended Rybnikov ones.

4.1. **Configurations of real points.**

For P, Q points in $\mathbb{RP}^2$, we denote by (P, Q) the line passing through P and Q.

Definition 4.1. A (t, m)-*configuration* $\mathcal{K}$ is the data $(\mathcal{V}, \mathcal{S}, \mathcal{L}, \mathrm{pl})$ composed of two finite sets of points $\mathcal{V} = \{V_1, \ldots, V_t\}$ and $\mathcal{S} = \{S_1, \ldots, S_n\}$ of $\mathbb{RP}^2$, the finite set of lines $\mathcal{L} = \{(S, V) \mid S \in \mathcal{S}, V \in \mathcal{V}\}$ in $\mathbb{RP}^2$ and a map $\mathrm{pl} : \mathcal{V} \sqcup \mathcal{S} \to \mathbb{Z}/m\mathbb{Z}$ with $\mathcal{V} = \mathrm{pl}^{-1}(0)$, such that:

(K1) for any $V_i, V_j \in \mathcal{V}$: $\mathcal{S} \cap (V_i, V_j) = \emptyset$,

(K2) for any line $L \in \mathcal{L}$: $\sum\limits_{S \in L \cap \mathcal{S}} \mathrm{pl}(S) = 0$.

The points in $\mathcal{V}$ (resp. in $\mathcal{S}$) are called the *vertices* (resp. *surrounding-points*) of $\mathcal{K}$. The map pl is called the m-*plumbing* of the configuration.

A (t, m)-configuration $\mathcal{K} = (\mathcal{V}, \mathcal{S}, \mathcal{L}, \mathrm{pl})$ is called *planar* if the projective subspace generated by the vertices $\mathcal{V}$ is the whole $\mathbb{RP}^2$, i.e. there exists three non-collinear points in $\mathcal{V}$. It is *uniform* if its plumbing map pl is constant on $\mathcal{S}$, i.e. there exists an element $\zeta \in \mathbb{Z}/m\mathbb{Z}$ such that $\mathrm{pl}(\mathcal{S}) = \{\zeta\}$. Note that any $(t, 2)$-configuration is necessarily uniform.

Example 4.2. Examples of $(3, 2)$, $(3, m)$ and $(4, 2)$-configurations are given in Figure (3). Remark that the dashed lines in the figures are not elements of $\mathcal{L}$, but they take an important role in our setting, as we show in Section 4.3.

In a similar manner as for arrangement, one can define the combinatorics of a (t, m)-configuration using non-trivial collinearity between points.

Definition 4.3. The *combinatorics* of a (t, m)-configuration $\mathcal{K} = (\mathcal{V}, \mathcal{S}, \mathcal{L}, \mathrm{pl})$ is the collection of all triples of collinear points in $\mathcal{V} \sqcup \mathcal{S}$.

To simplify the notation and to fit with the one of arrangements, if $k \geq 4$ different points $P_1, \ldots, P_k$ in $\mathcal{V} \sqcup \mathcal{S}$ are collinear, we write the set $\{P_1, \ldots, P_k\}$ instead of all the triples contained in $\{P_1, \ldots, P_k\}$. We say that two (t, m)-configurations $\mathcal{K}_1 = (\mathcal{V}_1, \mathcal{S}_1, \mathcal{L}_1, \mathrm{pl}_1)$ and $\mathcal{K}_2 = (\mathcal{V}_2, \mathcal{S}_2, \mathcal{L}_2, \mathrm{pl}_2)$ *are combinatorially equivalent* if there exists a one-to-one correspondence between the sets $\mathcal{V}_1 \sqcup \mathcal{S}_1$ and $\mathcal{V}_2 \sqcup \mathcal{S}_2$ respecting collinearity relations.

Example 4.4. The combinatorics of the $(3, 2)$-configuration of Figure (3)-(A) is given by

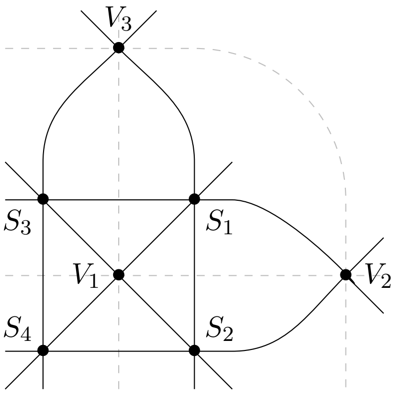
$$\{\ \{V_1, S_1, S_4\},\ \{V_1, S_2, S_3\},\ \{V_2, S_1, S_3\},\ \{V_2, S_2, S_4\},\ \{V_3, S_1, S_2\},\ \{V_3, S_3, S_4\}\ \}.$$

Remark 4.5. The combinatorics of a (t, m)-configuration is not invariant by isotopy. It is possible to create an extra alignment of points during a deformation, see Section 4.5.

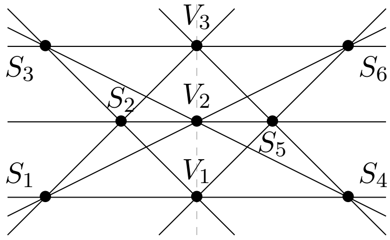
Definition 4.6. Let $\mathcal{K} = (\mathcal{V}, \mathcal{S}, \mathcal{L}, \mathrm{pl})$ be a (t, m)-configuration, and let $\mathcal{C}(\mathcal{K})$ be the combinatorics of $\mathcal{K}$.

- An *automorphism of* $\mathcal{C}(\mathcal{K})$ is a permutation ϕ of the set $\mathcal{V} \sqcup \mathcal{S}$ which preserves $\mathcal{C}(\mathcal{K})$. The group of such permutations is called the *automorphism group of* $\mathcal{C}(\mathcal{K})$ and is denoted by $\mathrm{Aut}(\mathcal{C}(\mathcal{K}))$.
- An automorphism of $\mathcal{C}(\mathcal{K})$ is *stabilizing* if $\phi(\mathcal{V}) = \mathcal{V}$ (equivalently, if $\phi(\mathcal{S}) = \mathcal{S}$). The subgroup of stabilizing automorphisms is denoted by $\mathrm{Aut}^{\mathrm{Stab}}(\mathcal{C}(\mathcal{K}))$.
- The configuration $\mathcal{K}$ is *stable* if $\mathrm{Aut}(\mathcal{C}(\mathcal{K})) = \mathrm{Aut}^{\mathrm{Stab}}(\mathcal{C}(\mathcal{K}))$.

Example 4.7. The $(3, 2)$-configuration pictured in Figure (3) (A) is stable, while the Pappus $(3, m)$-configuration in Figure (3) (D) is not.

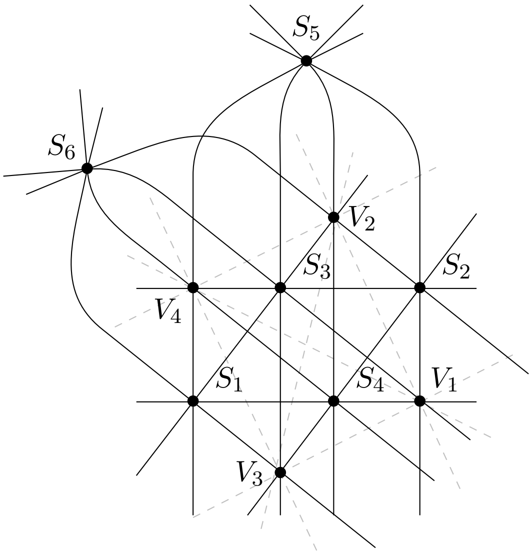


(A) Uniform $(3, 2)$-configuration.

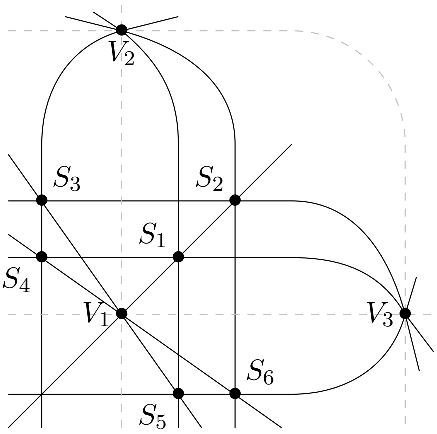


$\mathrm{pl} : (S_1, \cdots, S_6) \to (\zeta, -\zeta, \zeta, -\zeta, \zeta, -\zeta)$

(B) Non-uniform, non-planar $(3, m)$-configuration ($\zeta \in \mathbb{Z}/m\mathbb{Z}$).



(C) Uniform $(4, 2)$-configuration.



$\mathrm{pl} : (S_1, \ldots, S_6) \to (\zeta, -\zeta, \zeta, -\zeta, -\zeta, \zeta)$

(D) Pappus $(3, m)$-configuration ($\zeta \in \mathbb{Z}/m\mathbb{Z}$).

FIGURE 3. Examples of (t, m)-configurations

4.2. **Dual arrangements.**

Consider $\check{\mathbb{RP}}^2$ the *dual projective space* form by the lines of $\mathbb{RP}^2$. It is naturally isomorphic to the set of $\mathbb{R}$-linear forms in $\mathbb{R}^3$ modulo non-zero scalars. This duality between $\mathbb{RP}^2$ and $\check{\mathbb{RP}}^2$ respects the incidence relations, i.e. $P \in \ell$ if and only if $\ell^* \in P^*$. Note that, for any point P and any line ℓ, we have $(P^*)^* = P$ and $(\ell^*)^* = \ell$. To simplify the notation, if P is a point of $\mathbb{RP}^2$, we denoted by $P^\bullet$ the complex line of $\mathbb{CP}^2$ defined by $P^* \otimes \mathbb{C}$. For any set of points $\mathcal{P}$ in $\mathbb{RP}^2$, we denote be $\mathcal{A}^{\mathcal{P}}$ the dual arrangement defined by $\{P^\bullet \mid \forall P \in \mathcal{P}\}$.

Definition 4.8. Let $\mathcal{K} = (\mathcal{V}, \mathcal{S}, \mathcal{L}, \mathrm{pl})$ be a (t, m)-configuration.

1. The *dual arrangement* associated to $\mathcal{K}$ is the line arrangement $\mathcal{A}^{\mathcal{V} \sqcup \mathcal{S}}$.
2. The *dual character* of the plumbing pl is a torsion character ξ^{pl} of $\mathcal{A}^{\mathcal{K}}$ which assigns 1 to any line of $\mathcal{A}^{\mathcal{V}}$ and $\exp\big(2\pi i\, \mathrm{pl}(S)/m\big)$ to $S^\bullet \in \mathcal{A}^{\mathcal{S}}$.

Remark 4.9. By construction, the dual arrangement $\mathcal{A}^{\mathcal{K}}$ of any (t, m)-configuration $\mathcal{K}$ is a complexified real arrangement.

Proposition 4.10. *A quadruple $\mathcal{K} = (\mathcal{V}, \mathcal{S}, \mathcal{L}, \mathrm{pl})$ is a planar $(3, m)$-configuration if and only if $(\mathcal{A}^{\mathcal{K}}, \xi^{\mathrm{pl}}, \gamma)$ is a triangular inner-cyclic triple, where γ is a generator of $\gamma \in \mathrm{H}_1(\Gamma(\mathcal{A}^{\mathcal{V}}))$.*

In other words, $(3, m)$-configurations are dual of triangular inner-cyclic triples.

Idea of the proof. The arrangement $\mathcal{A}^{\mathcal{K}}$ and the character ξ^{pl} are the dual of the set of points $\mathcal{V} \sqcup \mathcal{S}$ and of the map pl respectively. Then Conditions (K1) and (K2) in Definition 4.1 correspond to Conditions (ICT1) and (ICT2) in Definition 2.1 respectively, in the particular case of triangular inner-cyclic triple. □

4.3. **The chamber weight of configurations.**

Proposition 4.10 gives a combinatorial equivalence between planar $(3, m)$-configurations and triangular inner-cyclic arrangements. We aim to pursue this analogy. To reach this goal, we introduce the *chamber weight* of a $(3, m)$-configuration, which is the dual version of the $\mathcal{I}$-invariant.

Let $\mathcal{K} = (\mathcal{V}, \mathcal{S}, \mathcal{L}, \mathrm{pl})$ be a planar $(3, m)$-configuration. The lines (V_1, V_2), (V_2, V_3) and (V_3, V_1) divide $\mathbb{RP}^2$ in 4 chambers, noted $\mathrm{ch}_1, \dots, \mathrm{ch}_4$. For any chamber ch_i, we define the value

$$\tau_i(\mathcal{K}) = \sum_{S \in \mathcal{S} \cap \mathrm{ch}_i} \mathrm{pl}(S) \in \mathbb{Z}/m\mathbb{Z}.$$

Proposition 4.11. *The value $\tau_i(\mathcal{K})$ is independent of i, i.e. $\tau_i(\mathcal{K}) = \tau_j(\mathcal{K})$ for all $i, j \in \{1, 2, 3, 4\}$. Moreover, it takes values over $\left\{[0], \left[\frac{m}{2}\right]\right\} \subset \mathbb{Z}/m\mathbb{Z}$ if m is even, and it is zero if m is odd.*

Definition 4.12. The *chamber weight* of $\mathcal{K}$ is

$$\tau(\mathcal{K}) = \sum_{S \in \mathcal{S} \cap \mathrm{ch}_i} \mathrm{pl}(S) \in \mathbb{Z}/m\mathbb{Z}.$$

Remark 4.13. In the case of a $(3, 2)$-configuration, $\tau(\mathcal{K})$ is the parity of the number of points of $\mathcal{S}$ contained in a single chamber ch_i.

Example 4.14. In Figure (3)–(A), we have one point S_i in each chamber ch_i. Since the configuration is uniform and $\mathrm{pl}(\mathcal{S}) \subset \mathbb{Z}/2\mathbb{Z} \setminus \{0\}$, then the chamber weight is

$$\tau(\mathcal{K}) = \mathrm{pl}(S_i) = 1.$$

Theorem 4.15. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two planar $(3, m)$-configurations. Assume that there exists a homeomorphism between $M(\mathcal{A}^{\mathcal{K}_1})$ and $M(\mathcal{A}^{\mathcal{K}_2})$ which respects fix orders on the associated arrangements. One has that:*

$$\tau(\mathcal{K}_1) = \tau(\mathcal{K}_2).$$

Idea of the proof. By Proposition 4.10, we have an equivalence between a $(3, m)$-configuration $\mathcal{K} = (\mathcal{V}, \mathcal{S}, \mathcal{L}, \mathrm{pl})$ and the triangular inner-cyclic triple $(\mathcal{A}^{\mathcal{K}}, \xi^{\mathrm{pl}}, \gamma)$. Since $\mathcal{A}^{\mathcal{K}}$ is a complexified real arrangement, then its braided wiring diagram can be given by the real picture. In particular, this implies that any intermediate braid b_i in the description (WD) is trivial.

Let denote by D_1, D_2 and D_3 the three lines of $\mathcal{A}^{\mathcal{V}}$, and assume that D_1 is the line at infinity. So that we can consider the affine picture of $\mathcal{A}^{\mathcal{K}} \setminus \{D_1\}$. By Theorem 2.9, one has that

$$\mathcal{I}(\mathcal{A}^{\mathcal{K}}, \xi^{\mathrm{pl}}, \gamma) = \xi^{\mathrm{pl}}(\mathrm{ulk}_{D_3} B_{(P \to D_3)} - \mathrm{ulk}_{D_2} B_{(P \to D_2)}),$$

where $P = D_2 \cap D_3$. It appears that $\mathrm{ulk}_{D_3} B_{(P \to D_3)} - \mathrm{ulk}_{D_2} B_{(P \to D_2)}$ is the sum of the meridians of the lines $D \in \mathcal{A}^{\mathcal{S}}$ such that the slope of their affine images in $\mathbb{RP}^2 \setminus D_1$ is upper and lower bounded by the slopes of D_2 and D_3, and the intersection $D \cap D_2$ is contained in the half plan $\{(x, y) \in \mathbb{R}^2 \mid x < x_P\}$, where x_P is the first coordinate of P in $\mathbb{R}^2$.

From the configuration's perspective, the two conditions above are similar. Let $\mathfrak{c}_i$ be one of the two cones in $\mathbb{RP}^2$ formed by (V_i, V_j) and (V_i, V_k), with $\{i,j,k\} = \{1,2,3\}$. The first and the second conditions are equivalent to $D^* \in \mathfrak{c}_1$ and $D^* \in \mathfrak{c}_3$, respectively. This corresponds to the fact that the dual point of D is in a fixed chamber τ_i, see Figure (4).

From the definition of the chamber weight, we obtain that:

$$\mathcal{I}(\mathcal{A}^{\mathcal{K}}, \xi^{\mathrm{pl}}, \gamma) = \exp(\frac{-2i\pi}{m}\tau(\mathcal{K})).$$

The result follows from Theorem 2.3. $\square$

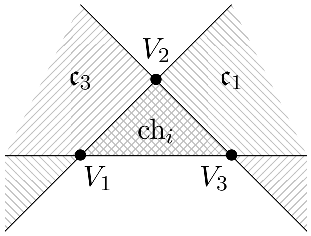


FIGURE 4. Intersection of two cones.

The preceding theorem can be enhanced by eliminating the ordered assumption. It is achieved if the value of the chamber weight is invariant under any automorphism of the combinatorics. While we might assume that the only automorphism of the combinatorics is the identity, it suffices to rely on a milder assumption: the stability of the configuration.

Corollary 4.16. *Let $\mathcal{K}$ be a stable planar uniform $(3,m)$-configuration, and consider $\mathcal{A}^{\mathcal{K}}$ as a non-ordered arrangement. The chamber weight $\tau(\mathcal{K})$ is an invariant of the homeomorphism type of $M(\mathcal{A}^{\mathcal{K}})$.*

4.4. **A Zariski pair of complexified real arrangements.**

We construct four $(3,2)$-configurations $\mathcal{K}_{1,1}, \mathcal{K}_{1,-1}, \mathcal{K}_{-1,1}$ and $\mathcal{K}_{-1,-1}$ which are combinatorially equivalent and verify that $\tau(C_{\alpha,\beta}) \neq \tau(C_{\alpha',\beta'})$ if $\alpha\beta \neq \alpha'\beta'$. Using Theorem 4.15 or Corollary 4.16, we conclude that the associated dual arrangements have non-homeomorphic complements.

For any $\alpha, \beta \in \{-1,1\}$, let $\mathcal{K}_{\alpha,\beta} = (\mathcal{V}, \mathcal{S}_{\alpha,\beta}, \mathcal{L}_{\alpha,\beta}, \mathrm{pl})$ be four uniform (3,2)-configurations defined by the following data

$$\mathcal{V} = \{V_1, V_2, V_3\}, \quad \mathcal{S}_{\alpha,\beta} = \{S_1, \dots, S_4\} \sqcup \{S_5^\alpha, S_6^\alpha, S_7^\alpha\} \sqcup \{S_8^\beta, S_9^\beta, S_{10}^\beta\},$$

where

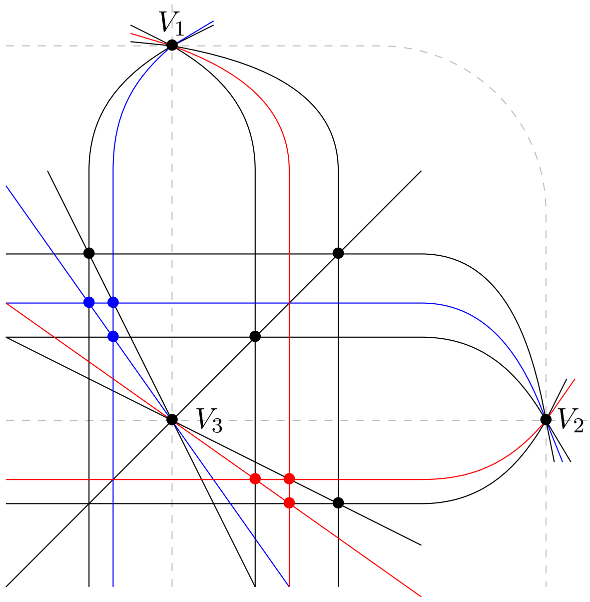
$$V_1 = (0:1:0), \quad V_2 = (1:0:0), \quad V_3 = (0:0:1),$$
$$S_1 = (1:1:1), \quad S_2 = (4:4:1), \quad S_3 = (-1:8:2), \quad S_4 = (8:-1:2),$$
$$S_5^\alpha = (-1:8:4\alpha), \quad S_6^\alpha = (-1:4\alpha:2), \quad S_7^\alpha = (-\alpha:4:4),$$
$$S_8^\beta = (8:-1:4\beta), \quad S_9^\beta = (4\beta:-1:2), \quad S_{10}^\beta = (4:-\beta:4).$$

Note that, for any $\alpha, \beta \in \{-1,1\}$, each line of $\mathcal{L}_{\alpha,\beta}$ contains exactly two surrounding-points. This is compatible with the 2-plumbing

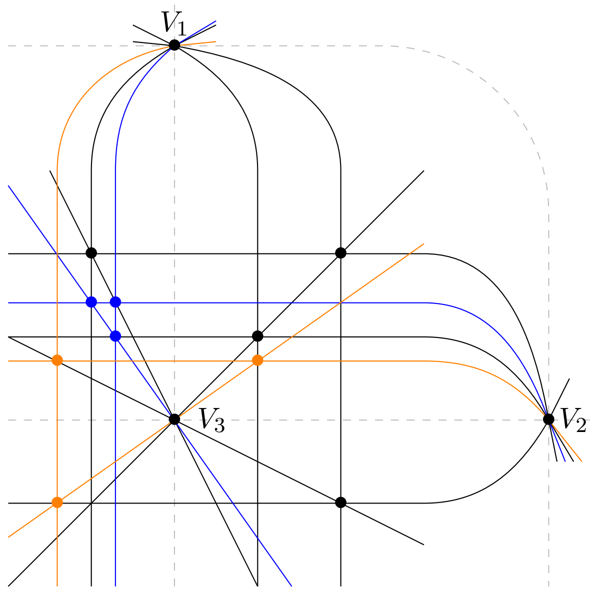
$$\mathrm{pl}(V) = 0 \in \mathbb{Z}/2\mathbb{Z}, \quad \forall V \in \mathcal{V} \quad \text{and} \quad \mathrm{pl}(S) = 1 \in \mathbb{Z}/2\mathbb{Z}, \quad \forall S \in \mathcal{S}_{\alpha,\beta}.$$

These four $(3,2)$-configurations are plotted[13] in Figure (5).

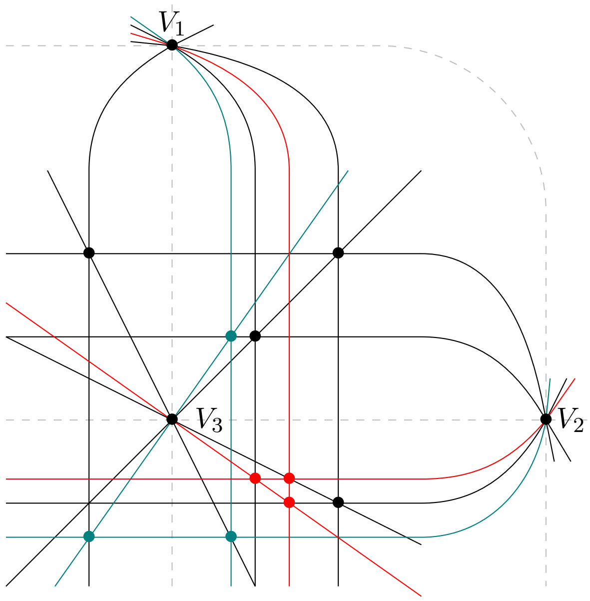
[13]To have clearer pictures, they are not plotted to scale but up to deformation respecting the combinatorics.



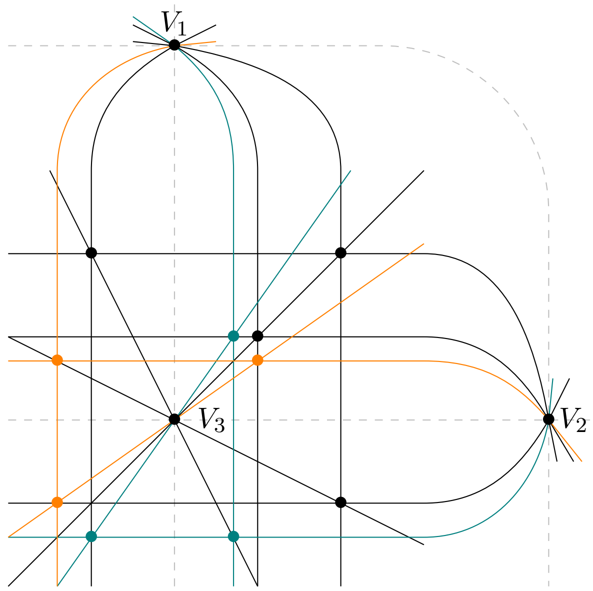
(A) The $(3,2)$-configuration $\mathcal{K}_{1,1}$.



(B) The $(3,2)$-configuration $\mathcal{K}_{1,-1}$.



(C) The $(3,2)$-configuration $\mathcal{K}_{-1,1}$.



(D) The $(3,2)$-configuration $\mathcal{K}_{-1,-1}$.

FIGURE 5. The $(3,2)$-configurations $\mathcal{K}_{\alpha,\beta}$. In black, the common points $\mathcal{V}$, $\mathcal{S}$ and their lines. In color, the points and lines corresponding to each configuration for $\alpha = 1$, $\alpha = -1$, $\beta = 1$, $\beta = -1$.

Proposition 4.17. *For any $\alpha, \beta \in \{-1, 1\}$, the configuration $\mathcal{K}_{\alpha,\beta}$ is stable and has the following combinatorics*

$$\left\{ \begin{array}{l} \{V_1, S_1, S_{10}^{\beta}\},\ \{V_1, S_2, S_4\},\ \{V_1, S_3, S_6^{\alpha}\},\ \{V_1, S_5^{\alpha}, S_7^{\alpha}\},\ \{V_1, S_8^{\beta}, S_9^{\beta}\} \\ \{V_2, S_1, S_7^{\alpha}\},\ \{V_2, S_2, S_3\},\ \{V_2, S_4, S_9^{\beta}\},\ \{V_2, S_5^{\alpha}, S_6^{\alpha}\},\ \{V_2, S_8^{\beta}, S_{10}^{\beta}\} \\ \{V_3, S_1, S_2\},\ \{V_3, S_3, S_5^{\alpha}\},\ \{V_3, S_4, S_8^{\beta}\},\ \{V_3, S_6^{\alpha}, S_7^{\alpha}\},\ \{V_3, S_9^{\beta}, S_{10}^{\beta}\} \end{array} \right\}.$$

For any $\alpha, \beta \in \{-1, 1\}$, we denote by $\mathcal{A}_{\alpha,\beta}$ the dual arrangement of the $(3,2)$-configuration $\mathcal{K}_{\alpha,\beta}$. Recall that, from Proposition 4.17, $\mathcal{A}_{1,1}$, $\mathcal{A}_{1,-1}$, $\mathcal{A}_{-1,1}$, $\mathcal{A}_{-1,-1}$ are combinatorially equivalent.

Theorem 4.18. *Let $\alpha, \alpha', \beta, \beta' \in \{-1, 1\}$ be such that $\alpha\beta \neq \alpha'\beta'$ and consider the arrangements $\mathcal{A}_{\alpha,\beta}$ and $\mathcal{A}_{\alpha',\beta'}$ as non-ordered arrangements. There is no homeomorphism between $M(\mathcal{A}_{\alpha,\beta})$ and $M(\mathcal{A}_{\alpha',\beta'})$.*

Proof. According to Corollary 4.16, it is sufficient to determine the parity of the number of surrounding-points in a chamber of the associated $(3,2)$-configuration. If we choose the bounded chambers in Figure (6), then one has:

$$\tau(\mathcal{K}_{1,1}) = [2] = [0], \quad \tau(\mathcal{K}_{1,-1}) = [3] = [1], \quad \tau(\mathcal{K}_{-1,1}) = [3] = [1] \quad \text{and} \quad \tau(\mathcal{K}_{-1,-1}) = [4] = [0]. \qquad \square$$

Corollary 4.19. *For any $\alpha, \beta, \alpha', \beta' \in \{-1, 1\}$ such that $\alpha\beta \neq \alpha'\beta'$, the non-ordered arrangements $\mathcal{A}_{\alpha,\beta}$ and $\mathcal{A}_{\alpha',\beta'}$ form a Zariski pair.*

It is worth pointing out again that the above result provides the first example of Zariski pair of line arrangements defined by rational coefficients and for which computer assistance isn't needed.

4.5. **A degeneration with 2 points of multiplicity 5.**

The moduli space of the previous examples is 3-dimensional. Indeed, if we fix by the action of $\mathrm{PGL}_3(\mathbb{C})$, the points V_1, V_2, V_3 and S_1, then the points S_2, S_3 and S_4 have each 1 degree of freedom. It appears that moving the points S_3 and S_4 to specific limit cases, we can create two alignments of 5 points: $V_1, S_1, S_3, S_6^\alpha, S_{10}^\beta$ in a first time, and $V_2, S_1, S_4, S_7^\alpha, S_9^\beta$ in a second one. The configurations then obtained are pictured in Figure (6), for $(\alpha, \beta) = (1, 1)$ and $(\alpha, \beta) = (1, -1)$. They are defined by $\mathcal{K}^2_{\alpha,\beta} = (\mathcal{V}^2, \mathcal{S}^2_{\alpha,\beta}, \mathcal{L}^2_{\alpha,\beta}, \mathrm{pl})$, where $\mathcal{V}^2 = \mathcal{V} = \{V_1, V_2, V_3\}$, with

$$V_1 = (0:1:0), \quad V_2 = (1:0:0), \quad V_3 = (0:0:1),$$

and $\mathcal{S}^2_{\alpha,\beta} = \{S_1, S_2, S_3, S_4\} \sqcup \{S_5^\alpha, S_6^\alpha, S_7^\alpha\} \sqcup \left\{S_8^\beta, S_9^\beta, S_{10}^\beta\right\}$ with

$$S_1 = (1:1:1), \quad S_2 = (4:4:1), \quad S_3 = (1:4:1), \quad S_4 = (4:1:1),$$

$$S_5^\alpha = (1:4:2\alpha), \quad S_6^\alpha = (1:2\alpha:1), \quad S_7^\alpha = (\alpha:2:2),$$

$$S_8^\beta = (4:1:2\beta), \quad S_9^\beta = (2\beta:1:1), \quad S_{10}^\beta = (2:\beta:2).$$

Remark 4.20. It is possible to move only one of the points S_3 and S_4 and create a unique alignment of 5 points. This construction is done in [48, Section 3.1]. The exponent 2 in the notation notifies the presence of the 2 alignments of 5 points. We decided to present this deformation due to additional properties on their fundamental groups that we explore in Section 5.1.

Proposition 4.21. *For any $\alpha, \beta \in \{-1, 1\}$, the $(3,2)$-configuration $\mathcal{K}^2_{\alpha,\beta}$ is stable and has the following combinatorics*

$$\left\{ \begin{array}{c} \{V_1, S_1, S_3, S_6^\alpha, S_{10}^\beta\}, \ \{V_1, S_2, S_4\}, \ \{V_1, S_5^\alpha, S_7^\alpha\}, \ \{V_1, S_8^\beta, S_9^\beta\} \\ \{V_2, S_1, S_4, S_7^\alpha, S_9^\beta\}, \ \{V_2, S_2, S_3\}, \ \{V_2, S_5^\alpha, S_6^\alpha\}, \ \{V_2, S_8^\beta, S_{10}^\beta\} \\ \{V_3, S_1, S_2\}, \ \{V_3, S_3, S_5^\alpha\}, \ \{V_3, S_4, S_8^\beta\}, \ \{V_3, S_6^\alpha, S_7^\alpha\}, \ \{V_3, S_9^\beta, S_{10}^\beta\} \end{array} \right\}.$$

We denote by $\mathcal{A}^2_{\alpha,\beta}$ the respective dual arrangements of $\mathcal{K}^2_{\alpha,\beta}$. By a computation of the chamber weight τ for these four configurations, we obtain the following theorem.

Theorem 4.22. *Let $\alpha, \beta, \alpha', \beta' \in \{-1, 1\}$ be such that $\alpha\beta \neq \alpha'\beta'$, and consider the arrangements $\mathcal{A}^2_{\alpha,\beta}$ and $\mathcal{A}^2_{\alpha',\beta'}$ as non-ordered. There is no homeomorphism between $M(\mathcal{A}^2_{\alpha,\beta})$ and $M(\mathcal{A}^2_{\alpha',\beta'})$.*

As a consequence, $\mathcal{A}^2_{\alpha,\beta}$ and $\mathcal{A}^2_{\alpha',\beta'}$ form a Zariski pair.

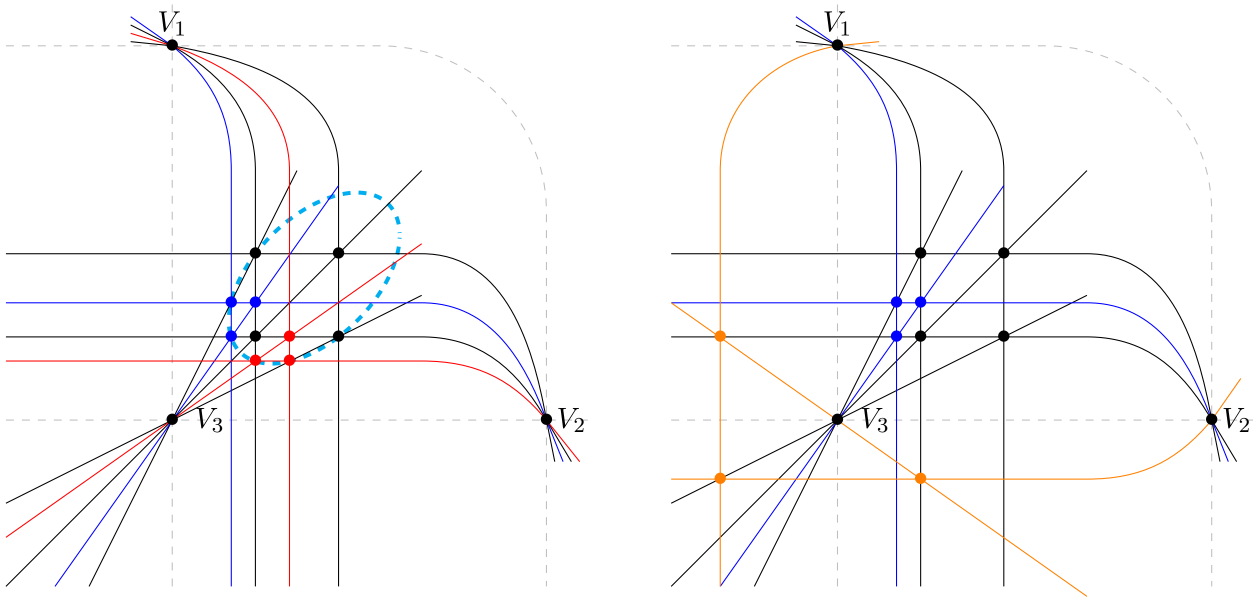


FIGURE 6. The $(3,2)$-configurations $\mathcal{K}^2_{1,1}$ and $\mathcal{K}^2_{1,-1}$, as well as the conic joining the six points $S_3, S_4, S_5^+, S_7^+, S_8^+, S_{10}^+$. In black, the common points $\mathcal{V}^2$, $\mathcal{S}^2$ and their lines. In color, the points and lines corresponding to each configuration for $\alpha = 1$, $\beta = 1$, $\beta = -1$.

4.6. **Moduli spaces & geometrical characterizations.**

In this section, we are interested in the moduli space of realizations of the Zariski pair given by $\mathcal{A}^2_{\alpha,\beta}$, presented in Section 4.5. In particular, in the spirit of Zariski in [88, 89, 90] (singular points on a conic) or of Artal in [5] (collinear points), geometric characterizations of the connected components of the moduli space are obtained.

4.6.1. *Moduli space of $\mathcal{A}^2_{\alpha,\beta}$.*

To describe an element of the moduli space, which is a class of arrangements projectively equivalent, one can give a representative of this class where four lines in generic position are fixed using the action of $\mathrm{PGL}_3(\mathbb{C})$. In the following theorem, the fixed lines are ℓ_1, ℓ_2, ℓ_3 and ℓ_4. To lighten the notation, the moduli space $\mathcal{M}(\mathcal{C}(\mathcal{A}^2_{\alpha,\beta}))$ is denoted by $\mathcal{M}^2$, since it is independent of α and β.

Theorem 4.23. *The moduli space $\mathcal{M}_2$ of $\mathcal{A}^2_{\alpha,\beta}$ is formed by the arrangements composed of the lines*

$$\ell_1 : y = 0, \quad \ell_2 : x = 0, \quad \ell_3 : z = 0, \quad \ell_4 : x + y + z = 0,$$

$$\ell_5 : \gamma x + \gamma y + z = 0, \quad \ell_6 : x + \gamma y + z = 0, \quad \ell_7 : \gamma x + y + z = 0,$$

$$\ell_8 : \kappa_1^{-1} x + \kappa_1 y + z = 0, \quad \ell_9 : x + \kappa_1 y + z = 0, \quad \ell_{10} : \kappa_1^{-1} x + y + z = 0,$$

$$\ell_{11} : \kappa_2 x + \kappa_2^{-1} y + z = 0, \quad \ell_{12} : \kappa_2 x + y + z = 0, \quad \ell_{13} : x + \kappa_2^{-1} y + z = 0.$$

where $\kappa_1^2 = \kappa_2^2 = \gamma \in \mathbb{C}^*$ satisfying $\kappa_1^3, \kappa_2^3 \neq 1$, $\kappa_1\kappa_2 \neq 1$ and:

(1) if $\kappa_1 = \kappa_2$:
- $2\kappa_1^2 + \kappa_1 + 1 \neq 0$,
- $2\kappa_1^2 + 2\kappa_1 + 1 \neq 0$,
- $\kappa_1 \neq -1/2$,
- $\kappa_1^3 + 3\kappa_1^2 + 2\kappa_1 + 1 \neq 0$,
- $\kappa_1^3 + 2\kappa_1^2 + \kappa_1 + 1 \neq 0$;

(2) if $\kappa_1 = -\kappa_2$:
- $\kappa_1^3 + \kappa_1^2 + 1 \neq 0$,
- $\kappa_1^3 - \kappa_1^2 - 1 \neq 0$,
- $2\kappa_1^2 + 1 \neq 0$,
- $\kappa_1^3 + \kappa_1 - 1 \neq 0$,
- $\kappa_1^3 + \kappa_1 + 1 \neq 0$.

Corollary 4.24. *There is no arithmetic Zariski pair with the combinatorics $\mathcal{C}(\mathcal{A}^2_{\alpha,\beta})$.*

Even if there are no Galois-conjugated realizations, we observe a Rybnikov-like construction. As explained in Section 3.2, Rybnikov arrangements are constructed by gluing two MacLane arrangements. We then obtain four arrangements $\mathcal{R}_{\pm,\pm} = \mathcal{ML}_{\pm} \cup \mathcal{ML}_{\pm}$ and $\mathcal{R}_{\pm,\mp} = \mathcal{ML}_{\pm} \cup \mathcal{ML}_{\mp}$, where $\mathcal{ML}_{\pm}$ are the MacLane arrangements [56]. The complex conjugation acts naturally on the equations of these arrangements. When this action is applied on all the lines it sends $\mathcal{R}_{\pm,\pm}$ and $\mathcal{R}_{\pm,\mp}$ on $\mathcal{R}_{\mp,\mp}$ and $\mathcal{R}_{\mp,\pm}$, respectively. Nevertheless, one can also consider a partial action of the complex conjugation on the second copy of the MacLane arrangements. This action sends $\mathcal{R}_{\pm,\pm}$ and $\mathcal{R}_{\pm,\mp}$ on $\mathcal{R}_{\pm,\mp}$ and $\mathcal{R}_{\pm,\pm}$, respectively. One qualifies such arrangements *semi-arithmetic* pair.

Using Theorem 4.23 for $\gamma = 2$, we can show that there exist arrangements in $\mathcal{M}_2$ which form such a semi-arithmetic pair. Indeed, if $\gamma = 2$, then $\kappa_1^2 = \kappa_2^2 = 2$ and so the arrangements associated have equations defined over $\mathbb{Q}(\sqrt{2})$. More precisely, when $\kappa_1 = \sqrt{2}$, one can consider the two arrangements $\mathcal{A}_{\sqrt{2}}$ and $\mathcal{A}_{-\sqrt{2}}$ of $\mathcal{M}_2$ defined by the following equations:

$$\ell_1 : y = 0, \quad \ell_2 : x = 0, \quad \ell_3 : z = 0, \quad \ell_4 : x + y + z = 0,$$

$$\ell_5 : 2x + 2y + z = 0, \quad \ell_6 : x + 2y + z = 0, \quad \ell_7 : 2x + y + z = 0,$$

$$\ell_8 : x + 2y + \sqrt{2}z = 0, \quad \ell_9 : x + \sqrt{2}y + 1 = 0, \quad \ell_{10} : \sqrt{2}x + 2y + 2z = 0,$$

$$\ell_{11} : 2x + y \pm \sqrt{2}z = 0, \quad \ell_{12} : \pm\sqrt{2}x + y + z = 0, \quad \ell_{13} : 2x \pm \sqrt{2}y + 2z = 0.$$

Consider the Galois action of $\mathbb{Q}(\sqrt{2})$. When it is applied only on the line ℓ_{11}, ℓ_{12} and ℓ_{13}, it sends $\mathcal{A}_{\sqrt{2}}$ on $\mathcal{A}_{-\sqrt{2}}$, and we obtain that these arrangements form a semi-arithmetic pair.

Remark 4.25. The arrangements $\mathcal{A}_{\alpha,\beta}$ defined in Section 4.4 also admit a similar Rybnikov-like construction, and so, a semi-arithmetic structure.

4.6.2. *Topology of the moduli space.*

From Theorem 4.23, we can obtain additional results on the topology of the moduli space. In the first place, we compute the number of connected components of the space.

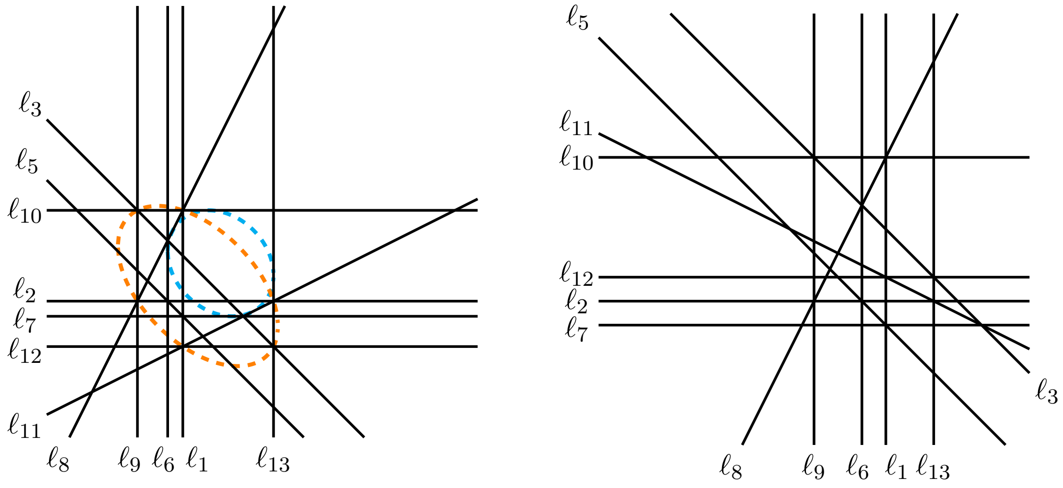
Proposition 4.26. *The moduli space $\mathcal{M}_2$ is formed by two connected components $\mathcal{M}_2^0$ and $\mathcal{M}_2^1$. The first is characterized by the relation $\kappa_1 = \kappa_2$, and the second by $\kappa_1 = -\kappa_2$.*

The previous result implies that $\mathcal{A}^2_{1,1}$ and $\mathcal{A}^2_{-1,-1}$ (resp. $\mathcal{A}^2_{-1,1}$ and $\mathcal{A}^2_{1,-1}$) are path-connected in $\mathcal{M}_2$, so by Randell Isotopy Theorem [70] they have the same topology. In addition, these two connected components are geometrically characterized. Let us denote by $P_{i_1,\dots,i_k}$ the singular point defined as the intersection of the lines $\ell_{i_1}, \dots, \ell_{i_k}$.

Proposition 4.27. *For any arrangement $\mathcal{A} \in \mathcal{M}_2$, the following are equivalent:*

(1) $\mathcal{A} \in \mathcal{M}_2^0$.
(2) The six lines $\ell_6, \ell_7, \ell_8, \ell_{10}, \ell_{11}, \ell_{13}$ are tangent to a smooth conic.
(3) The six triple points $P_{1,8,10}$, $P_{1,11,12}$, $P_{2,8,9}$, $P_{2,11,12}$, $P_{3,9,10}$ and $P_{3,12,13}$ are contained in a smooth conic.
(4) The three triple points $P_{1,11,12}$, $P_{2,8,9}$ and $P_{3,4,5}$ are collinear.

Remark 4.28. It can be checked that the characterizations given in Proposition 4.27 are not the only one of $\mathcal{M}_2^0$.



(A) A representative of $\mathcal{M}_2^0$ and the conics.

(B) A representative of $\mathcal{M}_2^1$.

FIGURE 7. Real representatives of the two connected components of $\mathcal{M}_2$, considering ℓ_4 as the line at the infinity.

5. Fundamental group of line arrangements

The fundamental group of the complement is a classical topological invariant of line arrangements. It has been known since Rybnikov [71, 72, 8] that it is not combinatorially determined. In this section, we present a complexified real pair of arrangements with non-isomorphic fundamental groups. They have been distinguished using the torsion of their Lower Central Series quotients. This solves at once the Falk and Randell Problem 1.3 in [36], and the Question 8.7 of Suciu in [78]:

Problem (Falk and Randell [36]). *Prove that the underlying matroid of a complexified arrangement determines the homotopy type, or find a counter-example.*

Question (Suciu [78]). *Let $G(\mathcal{A})$ be an arrangement group. Is the torsion in $\operatorname{gr}_k G(\mathcal{A})$ combinatorially determined?*

Then, on the other side of the spectrum, we construct examples of combinatorially and homotopy-equivalent arrangements with non-homeomorphic complements. This result first appears in [43] as the existence of homotopy-equivalent Zariski pairs. It is improved to non-homeomorphic complements in [45] using the multiplicativity theorem of the loop-linking number. This result completes the works of Falk [32], and Jiang and Yau [51, 53] on the relation between the homotopy of $M(\mathcal{A})$ and the combinatorics of the arrangement.

5.1. Complexified real arrangements & Fundamental group.

As a first step in the study of the fundamental group of complexified real arrangements, we need to compute $G(\mathcal{A}) = \pi_1(M(\mathcal{A}))$. In such case, Randell [68, 69] gives an algorithm to compute a finite presentation of $G(\mathcal{A})$ from the real picture of the arrangement, see also [73]. This has been generalized by Arvola in [14] to any complex line arrangement.

5.1.1. *Computation of the fundamental group.*

Throughout this section, we assume that $\mathcal{A} = \{\ell_0, \dots, \ell_n\}$ is a complexified real arrangement[14]. Let us recall how to compute the presentation of $G(\mathcal{A})$ given by Randell. We assume that ℓ_0 is the line

[14]The choice to start the indices at 0 and not 1 as above allows to take ℓ_0 as the line at infinity and consider the affine part of $\mathcal{A}$ and to have line labeled from 1 to n.

$z = 0$ and is considered as the line at infinity. Let $\mathcal{A}^{\mathbb{R}} = \mathcal{A} \cap \mathbb{R}^2$ be the real picture of $\mathcal{A}$. By an abuse of notation, the affine real trace of $\ell_i \in \mathcal{A}$ is also denoted by ℓ_i. Last, we assume that no line of $\mathcal{A}$ has a defining equation of the form $x = \alpha$. Assign to each line ℓ_i of $\mathcal{A}^{\mathbb{R}}$ a meridian $\mathfrak{m}_i$ contained in a complex line defined by $x = \beta$ with $\beta \in \mathbb{R}$, such that the first coordinate of each singular point of $\mathcal{A}$ is strictly greater than β.

Reading the picture $\mathcal{A}^{\mathbb{R}}$ from the negative part of the first coordinate of $\mathbb{R}^2$ to its positive part[15], we assign to each smooth part in $\mathcal{A}^{\mathbb{R}}$, i.e. the segments bounded by the singularities, a conjugate of the meridian of the associated line. This process is described in Figure (8).

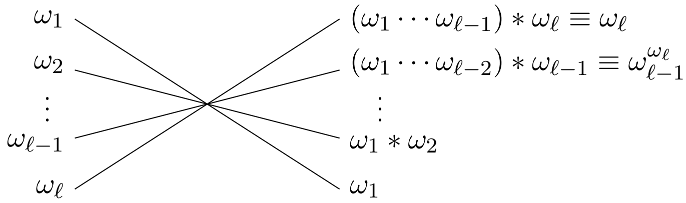


FIGURE 8. Assignment of meridians at each singularity.

Finally, to each singular point $P \in \mathrm{Sing}(\mathcal{A}^{\mathbb{R}})$ with input elements $\omega_1, \dots, \omega_\ell$ associated to the left-hand segments (as in Figure (8)), we assign the set of $m(P) - 1$ relations.

$$R_P = \{\omega_1 \cdots \omega_\ell = \omega_{\sigma(1)} \cdots \omega_{\sigma(\ell)} \mid \sigma \text{ a cyclic permutation of } \ell \text{ elements}\}$$

Remark 5.1. This method works even if there are vertical lines; by a small rotation, e.g. counterclockwise, one can assume that the vertical line is the first one. Examples for double and triple points are given in Figure (9).

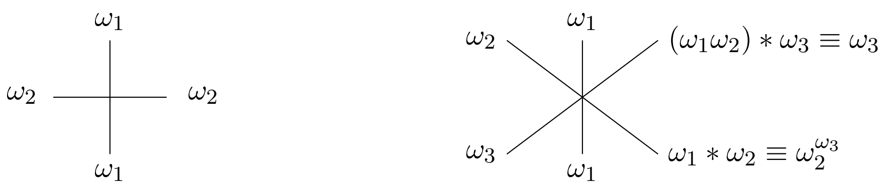


FIGURE 9. Assignments of meridians with vertical lines.

Theorem 5.2 (Randell [68, 69]). *Let $\mathcal{A}$ be a complexified real arrangement. The fundamental group of the complement $M(\mathcal{A})$ admits the following presentation:*

$$G(\mathcal{A}) \simeq \left\langle \; \mathfrak{m}_1, \dots, \mathfrak{m}_n \;\middle|\; \bigcup_{P \in \mathrm{Sing}(\mathcal{A}^{\mathbb{R}})} R_P \; \right\rangle.$$

5.1.2. *Lower central series quotients.*

The *lower central series* (LCS) of a group G is defined as a descending sequence of normal subgroups

$$G = \gamma_1(G) \trianglerighteq \gamma_2(G) \trianglerighteq \cdots \trianglerighteq \gamma_k(G) \trianglerighteq \cdots,$$

such that each $\gamma_{k+1}(G)$ is the commutator subgroup $[\gamma_k(G), G]$ of G. The *kth lower central quotient* of this series is defined as the group $\mathrm{gr}_k(G) = \gamma_k(G)/\gamma_{k+1}(G)$. Note that $\mathrm{gr}_k(G)$ is Abelian since $[\gamma_k(G), \gamma_k(G)] \leq \gamma_{k+1}(G)$. In addition, if G is finitely generated, then so $\mathrm{gr}_k(G)$ is. Thus, $\mathrm{gr}_k(G)$ is fully determined by its rank, noted $\phi_k(G)$, and its torsion.

[15]This direction of reading is taken from left to right in the different pictures of this manuscript.

Remark 5.3. If G is a finitely presented group, then the GAP package `nq` can be used to compute the lower central series quotients of G.

In the case of a group $G(\mathcal{A})$ associated to a line arrangement $\mathcal{A}$, the possible dependency on the combinatorics of the invariants of the LCS is a classical research topic. In [71, 72], Rybnikov gives a sketch of a proof of the combinatorial determination of the second nilpotent group $G(\mathcal{A})/\gamma_3(G(\mathcal{A}))$, lately formally proved by Matei and Suciu [58]. Using Sullivan 1-minimal models, Falk proves that also $\phi_k(G(\mathcal{A}))$ is combinatorially determined [31]. He also obtains with Randell [34], in the particular case of fiber-type arrangements, the *LCS formula*:

$$\prod_{k\geq 1}(1-t^k)^{\phi_k(G(\mathcal{A}))} = P(M(\mathcal{A}), -t),$$

where $P(M(\mathcal{A}), t)$ is the Poincaré polynomial of the complement (see [67, Sec. 2]), which is known to be combinatorial. Nevertheless, there exist examples of arrangements for which the LCS formula fails [36, 78], and no general explicit formula is known for $\phi_k(G(\mathcal{A}))$, even for $\phi_3(G(\mathcal{A}))$.

5.1.3. *Fundamental groups of complexified real arrangements are not combinatorially determined.*

Let $\mathcal{A}_{\sqrt{2}}$ and $\mathcal{A}_{-\sqrt{2}}$ be the two arrangements defined in Section 4.6.1. From Theorem 4.22, we know that their complements $M(\mathcal{A}_{\sqrt{2}})$ and $M(\mathcal{A}_{-\sqrt{2}})$ are not isomorphic. Consider $G(\mathcal{A}_{\sqrt{2}})$ and $G(\mathcal{A}_{-\sqrt{2}})$ the fundamental groups of $M(\mathcal{A}_{\sqrt{2}})$ and $M(\mathcal{A}_{-\sqrt{2}})$, respectively. Using the algorithm given in Section 5.1.1 and Figure (7), we can compute that in $G(\mathcal{A}_{\sqrt{2}})$ and $G(\mathcal{A}_{-\sqrt{2}})$, the torsion of the 4th and 5th LCS quotients differs. Indeed, using the `GAP` code described in the Appendix of [13], we can compute the primary decompositions of the five first lower central series quotients and then obtain the following theorem.

Theorem 5.4. *The first LCS quotients of $G(\mathcal{A}_{\sqrt{2}})$ and $G(\mathcal{A}_{-\sqrt{2}})$ have the following primary decompositions:*

(1) for $k\leq 3$, we have $\mathrm{gr}_k(G(\mathcal{A}_{\sqrt{2}}))\simeq \mathrm{gr}_k(G(\mathcal{A}_{-\sqrt{2}}))\simeq \mathbb{Z}^{\phi_k}$ with $(\phi_k)_{k=1}^3=(12,23,76)$,
(2) $\mathrm{gr}_4(G(\mathcal{A}_{\sqrt{2}}))\simeq \mathbb{Z}^{\phi_4}\oplus\mathbb{Z}_2$ and $\mathrm{gr}_4(G(\mathcal{A}_{-\sqrt{2}}))\simeq \mathbb{Z}^{\phi_4}$, with $\phi_4=211$,
(3) $\mathrm{gr}_5(G(\mathcal{A}_{\sqrt{2}}))\simeq \mathbb{Z}^{\phi_5}\oplus\mathbb{Z}_2$ and $\mathrm{gr}_5(G(\mathcal{A}_{-\sqrt{2}}))\simeq \mathbb{Z}^{\phi_5}$, with $\phi_5=660$.

Remark 5.5. We can show that the commutator of meridians $[[[\mathfrak{m}_1,\mathfrak{m}_5],\mathfrak{m}_2],\mathfrak{m}_3]$ is a representative of the 2-torsion element in $\mathrm{gr}_4(G(\mathcal{A}_{\sqrt{2}}))$ (using `GAP`), while it corresponds to the identity in $\mathrm{gr}_4(G(\mathcal{A}_{-\sqrt{2}}))$.

Corollary 5.6. *The torsion in the quotients of the LCS of $G(\mathcal{A})$ is not determined by the combinatorics $\mathcal{C}(\mathcal{A})$.*

Remark 5.7. It can be verified that the LCS formula fails for these arrangements.

Corollary 5.8. *The fundamental group of the complement of a complexified real arrangement $\mathcal{A}$ is not determined by its combinatorics $\mathcal{C}(\mathcal{A})$.*

Remark 5.9. Theorem 5.4 implies in particular that $\mathcal{A}_{\sqrt{2}}$ and $\mathcal{A}_{-\sqrt{2}}$ have non-equivalent braid monodromies.

5.2. **Homotopy equivalent Zariski pairs.**

In this section, we explore the other extremity of the range of Zariski pairs. Previously, we exhibited a Zariski pair with non-isomorphic fundamental groups. Here, we give examples of π_1-equivalent Zariski pairs [43], but also Zariski pairs with homotopically-equivalent and non-homeomorphic complements [43, 45]. These results take their inspiration from the intersection of the works of Rybnikov [71, 71], and Oka and Sakamoto [65]: the former for the construction of Zariski pairs using gluing of arrangements, the latter for the control of the fundamental groups.

5.2.1. *Augmented arrangement and homotopy of the complement.*

Let $\mathcal{A} = \{\ell_1, \dots, \ell_n\}$ be an ordered arrangement, and let ℓ be a fixed line of $\mathcal{A}$. An *augmented arrangement* of $\mathcal{A}$ along ℓ is an ordered arrangement $\mathcal{A}_\ell^+ = \{\ell_1, \dots, \ell_n, \ell_{n+1}, \ell_{n+2}\}$ such that:

(A1) the lines ℓ, ℓ_{n+1} and ℓ_{n+2} intersect a triple point of $\mathcal{A}_\ell^+$,

(A2) the arrangements $\{\ell_{n+1}, \ell_{n+2}\}$ and $\mathcal{A} \setminus \{\ell\}$ intersect generically, i.e. one has

$$\#\mathcal{Z}(\{\ell_{n+1}, \ell_{n+2}\}) \cap \mathcal{Z}(\mathcal{A} \setminus \{\ell\}) = 2n - 2.$$

This construction permits to keep a control on the homotopy of the complement of the augmented arrangement, as stated in the following theorem.

Theorem 5.10 ([65, 85])**.** *Let $\mathcal{A}$ be an ordered arrangement and ℓ, ℓ' be two lines of $\mathcal{A}$. The arrangements $\mathcal{A}_\ell^+$ and $\mathcal{A}_{\ell'}^+$ are π_1-equivalent. More precisely:*

$$\pi_1(M(\mathcal{A}_\ell^+)) \simeq \pi_1(M(\mathcal{A})) \times \mathbb{F}_2 \simeq \pi_1(M(\mathcal{A}_{\ell'}^+)).$$

Furthermore, if $\mathcal{A}$ is a complexified real arrangement, then the complement of $\mathcal{A}_\ell^+$ and $\mathcal{A}_{\ell'}^+$ are homotopy-equivalent.

The first part of the previous theorem is due to Oka and Sakamoto in [65]. Notice that this can also be obtained from [33] and [50]. Furthermore, since an augmented arrangement $\mathcal{A}_\ell^+$ is nothing else than a 2-generic section of the parallel connection of the arrangement $\mathcal{A}$ with a pencil of 3 lines (see [33, 26] for more details about parallel connections), then the second part of the theorem is given by Williams in [85].

5.2.2. *Homotopy-equivalent and π_1-equivalent Zariski pairs.*

Let $\mathcal{A}_1 = \{\ell_1^1, \dots, \ell_n^1\}$ and $\mathcal{A}_2 = \{\ell_1^2, \dots, \ell_m^2\}$ be two ordered arrangements intersecting generically. We denote the ordered arrangement[16] $\mathcal{A}_1 \sqcup \mathcal{A}_2$ by $\mathcal{A}_{1,2}$ where the order is given by:

$$(\ell_1^1, \dots, \ell_n^1, \ell_1^2, \dots, \ell_m^2) \longmapsto (1, \dots, n+m).$$

Remark 5.11. The arrangements $\mathcal{A}_{1,2}$ and $\mathcal{A}_{2,1}$ are not the same ordered arrangement even if they are the same non-ordered arrangement.

To prove our main result, let us recall the notion of *connected arrangement* introduced by Fan [37]. Let $\mathrm{Sing}_{\geq 3}(\mathcal{A})$ and $\mathrm{Sing}_2(\mathcal{A})$ be the subsets of $\mathrm{Sing}(\mathcal{A})$ form respectively by the multiple points and the double points of $\mathcal{A}$. An arrangement $\mathcal{A}$ is connected if the set $\mathcal{A}_{\geq 3} = \mathcal{Z}(\mathcal{A}) \setminus \mathrm{Sing}_2(\mathcal{A})$ is path-connected. Note that this property is combinatorial.

Theorem 5.12. *Let $\mathcal{A}_1 = \{\ell_1^1, \dots, \ell_n^1\}$ and $\mathcal{A}_2 = \{\ell_1^2, \dots, \ell_n^2\}$ be a Zariski pair, ϕ be the ordered isomorphism between their combinatorics, i.e. $\phi(\ell_i^1) = \ell_i^2$. We fix $k \in \{1, \dots, n\}$ and denote by ℓ_j the line ℓ_k^j, for $j \in \{1, 2\}$. We assume that:*

(C1) The arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ are connected,

(C2) They intersect generically,

(C3) For $j \in \{1, 2\}$, any line ℓ of $\mathcal{A}_j$ contains at least two multiple points, i.e. $|\ell \cap \mathrm{Sing}_{\geq 3}(\mathcal{A}_j)| \geq 2$.

The arrangements $(\mathcal{A}_{1,2})_{\ell_1}^+$ and $(\mathcal{A}_{2,1})_{\ell_2}^+$ verify the following properties:

(P1) They have isomorphic intersection lattices,

(P2) There is no homeomorphism of $\mathbb{CP}^2$ sending $(\mathcal{A}_{1,2})_{\ell_1}^+$ on $(\mathcal{A}_{2,1})_{\ell_2}^+$,

[16] This construction of an ordered union of arrangements is similar to the one make in Section 3.1, the only difference is the genericity of the intersection.

(P3) Their complements are π_1-equivalent; furthermore, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are real-complexified arrangements then the complements are homotopy-equivalent.

Remark 5.13. The conditions (C1) and (C3) are combinatorial; thus if they are verified by $\mathcal{A}_1$ then they are also verified by $\mathcal{A}_2$; and, up to the action of $\mathrm{PGL}_3(\mathbb{C})$, condition (C2) is always true.

Proof.
• (P1): The application $\phi^+ : (\mathcal{A}_{1,2})^+_{\ell_1} \to (\mathcal{A}_{2,1})^+_{\ell_2}$ defined below is an (ordered) isomorphism between the intersection lattices.

$$\phi^+ : \begin{cases} \ell_i^1 & \longmapsto \quad \ell_i^2 \\ \ell_i^2 & \longmapsto \quad \ell_i^1 \\ \ell_{2n+1} & \longmapsto \quad \ell_{2n+1} \\ \ell_{2n+2} & \longmapsto \quad \ell_{2n+2} \end{cases}$$

• (P2): We assume that there exists a homeomorphism ψ^+ of $\mathbb{CP}^2$ sending $(\mathcal{A}_{1,2})^+_{\ell_1}$ on $(\mathcal{A}_{2,1})^+_{\ell_2}$. Condition (C3) implies that $\psi^+(\{\ell_{2n+1}, \ell_{2n+2}\}) = \{\ell_{2n+1}, \ell_{2n+2}\}$. Thus, ψ^+ is also a homeomorphism between $\mathcal{A}_{1,2}$ and $\mathcal{A}_{2,1}$. From Conditions (C1) and (C2), we deduce that ψ^+ fixes or exchanges $\mathcal{A}_1$ and $\mathcal{A}_2$. Finally, due to Condition (C3) and the definition of an augmented arrangement, we have that $\psi^+(\ell_1) = \ell_2$. This implies that ψ^+ sends $\mathcal{A}_1$ and $\mathcal{A}_2$, which is impossible since $\mathcal{A}_1$ and $\mathcal{A}_2$ form a Zariski pair.

• (P3): As non-ordered arrangements, $\mathcal{A}_{1,2}$ and $\mathcal{A}_{2,1}$ are the same arrangement. Thus, $(\mathcal{A}_{1,2})^+_{\ell_1}$ and $(\mathcal{A}_{2,1})^+_{\ell_2}$ are two augmentations of the same arrangement along different lines. We conclude using Theorem 5.10. ☐

The Zariski pairs given in [72, 41] verify Conditions (C1), (C2) and (C3) of Theorem 5.12. Furthermore, the ones given in [7, 48] are complexified real Zariski pairs, and also verify the conditions of Theorem 5.12. This allows to prove the following corollary.

Corollary 5.14. *For a fixed combinatorics, the topology of an arrangement $\mathcal{A}$ is determined neither by the fundamental group nor by the homotopy-type of its complement $M(\mathcal{A})$.*

5.2.3. *Improvement with non-homeomorphic complements.*

Using the multiplicativity Theorem for the $\mathcal{I}$-invariant (Theorem 3.1), one can improve Corollary 5.14 and replace the topology of an arrangement $\mathcal{A}$ by the homeomorphism type of its complement $M(\mathcal{A})$. In Section 3.1, we introduce the notation $\mathcal{A}_1 \triangle_\phi \mathcal{A}_2$ for the gluing of two arrangements according to ϕ. In this section, $\mathcal{A}_1 \triangle \mathcal{A}_2$ will denote any generic gluing of $\mathcal{A}_1$ with $\mathcal{A}_2$. Similarly to Remark 5.11, as non-ordered arrangements $\mathcal{A}_1 \triangle \mathcal{A}_2$ and $\mathcal{A}_2 \triangle \mathcal{A}_1$ are the same, although they are different ordered arrangements. In particular, they have isomorphic fundamental groups and equivalent non-ordered combinatorics.

Let $\mathcal{A}_1 = \{\ell_1^1, \dots, \ell_n^1\}$ and $\mathcal{A}_2 = \{\ell_1^2, \dots, \ell_n^2\}$ be two combinatorially equivalent arrangements, and denote by $\mathcal{C}$ their combinatorics. Assume that there exist ξ and γ respectively a character on $\mathcal{C}$ and a cycle of $\Gamma(\mathcal{C})$ such that $(\mathcal{A}_1, \xi, \gamma)$ and $(\mathcal{A}_2, \xi, \gamma)$ are triangular inner-cyclic triples[17] which verify:

$$\mathcal{I}(\mathcal{A}_1, \xi, \gamma) \neq \mathcal{I}(\mathcal{A}_2, \xi, \gamma)^{\pm 1}.$$

[17] Let us recall that according to the definition given in Section 3.1, the only line-vertices contained by γ are $v_{\ell_1^j}$, $v_{\ell_2^j}$ and $v_{\ell_3^j}$.

Last, assume that the value of the $\mathcal{I}$-invariant is independent of the orders considered on $\mathcal{A}_1$ and $\mathcal{A}_2$. In other words, $\mathcal{A}_1$ and $\mathcal{A}_2$ form a Zariski pair distinguished by the $\mathcal{I}$-invariant. Such arrangements are given in Section 2.3, and in Section 4.4 for the particular case of complexified real arrangements.

Let fix $k \in \{4, \dots, n\}$. For $j \in \{1, 2\}$, we denote by ℓ_j the line ℓ_k^j. Let $\mathfrak{A}_1 = \mathcal{A}_1 \triangle (\mathcal{A}_2)^+_{\ell_2}$ and $\mathfrak{A}_2 = \mathcal{A}_2 \triangle (\mathcal{A}_1)^+_{\ell_1}$. By construction, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are combinatorially equivalent. We denote by Id the trivial character on $(\mathcal{A}_j)^+_{\ell_j}$ which sends all the meridians on 1. Finally, let μ be the cycle of $\mathfrak{A}_j$ which contains as line-vertices only v_{ℓ_1}, v_{ℓ_2} and v_{ℓ_3}. Using the multiplicativity Theorem 3.1, one has:

$$\mathcal{I}(\mathfrak{A}_j, \xi \triangle \mathrm{Id}, \mu) = \mathcal{I}(\mathcal{A}_j, \xi, \gamma).$$

By Theorem 2.3, as non-ordered arrangements $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have non-homeomorphism complements. If $\mathcal{A}_1$ and $\mathcal{A}_2$ verify Conditions (C1) and (C3), then using the same argument as in the proof of Theorem 5.12 (P2), we obtain that $\mathcal{I}(\mathfrak{A}_j, \xi \triangle \mathrm{Id}, \mu)$ is also independent of the order of the combinatorics. At the end, one has the following:

Theorem 5.15. *The non-ordered arrangements $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have the following properties:*

1. *They are combinatorially equivalent.*
2. *Their complements are non-homeomorphic.*
3. *Their complements are π_1-equivalent.*
4. *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are complexified real arrangements, then their complements are homotopically equivalent.*

As noticed above, the arrangements are given in Section 2.3, and in Section 4.4 for the particular case of complexified real arrangements verify all the hypothesis. So one can deduce the following.

Corollary 5.16. *For a fixed combinatorics, the homeomorphism type of an arrangement complement $M(\mathcal{A})$ is determined neither by its fundamental group nor by its homotpy-type.*

Corollary 5.17. *The $\mathcal{I}$-invariant of an inner-cyclic triple $(\mathcal{A}, \xi, \gamma)$ is not determined by the homotopy type of the complement $M(\mathcal{A})$.*

5.2.4. *Explicit examples with less lines.*

One of the weakness of the previous construction is the number of lines needed to construct the examples. The smallest example constructed contains 21 lines, and 25 lines for the complexified real one. It appears that we decrease these numbers by considering an ordered Zariski pair which is not a Zariski pair.

Consider the arrangements $\mathcal{M}^{\pm}$ and $\mathcal{N}^{\pm}$ of 11 lines defined in Section 2.3. As we noticed in Remark 2.12, there exists a projective transformation that realizes the automorphism of the combinatorics. It cyclically permutes $\mathcal{M}^+$, $\mathcal{N}^+$, $\mathcal{M}^-$ and $\mathcal{N}^-$. This implies that they have the same topology and so they are π_1-equivalent. Nevertheless, we proved that they form ordered Zariski pairs, as soon as the two considered arrangements are not complex conjugated. In Section 2.3, we add a line to reduce the automorphism group of the combinatorics to the trivial group. It is proven, in [11], that this makes the fundamental groups non-isomorphic. In this section, we apply a different strategy to reduce the automorphism group to the trivial one.

By Proposition 2.11, we know that the automorphism group of the combinatorics $\mathcal{K}_{11}$ is generated by the permutation

$$\sigma = (\ell_1, \ell_2)(\ell_4, \ell_6, \ell_8, \ell_{10})(\ell_5, \ell_7, \ell_9, \ell_{11}).$$

The permutation σ acts cyclically on ℓ_4, ℓ_6, ℓ_8 and ℓ_{10}. By taking an augmentation of $\mathcal{M}^{\pm}$ and $\mathcal{N}^{\pm}$, we can make combinatorially unique one of them, ℓ_4 for example. This forces any homeomorphism between the arrangements to respect the order, and that creates a contradiction.

Theorem 5.18. *The arrangements $(\mathcal{M}^{\pm})^+_{\ell_4}$ and $(\mathcal{N}^{\pm})^+_{\ell_4}$ verify the following properties:*

(1) They are combinatorially equivalent, and their automorphism group if isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and permutes the two lines ℓ_{12} and ℓ_{13}.

(2) They are π_1-equivalent with fundamental group isomorphic to $\pi_1(M(\mathcal{M}^{\pm}))\times\mathbb{F}_2 \simeq \pi_1(M(\mathcal{N}^{\pm}))\times\mathbb{F}_2$.

(3) As non-ordered arrangements, they have non-homeomorphic complements.

In other words, the arrangements $(\mathcal{M}^{\pm})^+_{\ell_4}$ and $(\mathcal{N}^{\pm})^+_{\ell_4}$ form π_1-equivalent Zariski pair. Furthermore, they have only 13 lines.

To construct a smaller homotopy-equivalent Zariski pair, we apply a similar argument as previously to the example of Artal, Carmona, Cogolludo and Marco [7]. Unfortunately, using a single augmentation is not enough to fix all the automorphisms of the combinatorics as previously done. This problem can be avoided by the application of two successive augmentations.

Let a be a root of X^2+X-1, and consider the arrangements $\mathcal{M}$ and $\mathcal{N}$ formed by the 10 lines:

$$\begin{array}{ll}
M_1 : z = 0, & \ell_1 : x - y, \\
M_2 : x = 0, & \ell_2 : ax - y - az = 0, \\
M_3 : x - z = 0, & \ell_3 : ax - y + z = 0, \\
M_4 : x + (a+1)z = 0, & \ell_4 : y - z = 0, \\
M_5 : x - (a+2)z = 0, & \ell_5 : y = 0.
\end{array}$$

Theorem 5.19 (Remark 2.8 and Theorem 4.19 of [7])**.** *The arrangements $\mathcal{M}$ and $\mathcal{N}$ form a homotopy-equivalent ordered Zariski pair.*

By [7, Lemmas 2.4 & 2.9], the automorphism group of the combinatorics of $\mathcal{M}$ (and $\mathcal{N}$ too) is isomorphic to the subgroup of Σ_5 (the symmetric group on 5 elements) generated by:

$$\sigma_1 = (1,2,3,4,5) \quad \text{and} \quad \sigma_2 = (2,4,5,3).$$

More precisely, it is the semi-direct product of $\langle\sigma_1\rangle$ and $\langle\sigma_2\rangle$.

The action of the generators σ_1 and σ_2 can be viewed geometrically as an action on the lines M_i and ℓ_i which is given by $\sigma_j \cdot M_i = M_{\sigma_j(i)}$ and $\sigma_j \cdot \ell_i = \ell_{\sigma_j(i)}$. The idea is to trivialize this action by taking augmentations of these arrangements. The subtlety of this case lies in the following fact: if we combinatorially fix one of the line M_i or ℓ_i with an augmentation, then there are still some non-trivial automorphisms acting on M_i and ℓ_i. So, to trivialize this action, we need to consider two augmentations. The first fixes M_1 (and so ℓ_1), while the second fixes ℓ_5 (and so M_5).

Let $\mathfrak{M}^+_{M_1,\ell_5}$ (resp. $\mathfrak{N}^+_{M_1,\ell_5}$) be the arrangement arising from two augmentations of $\mathcal{M}$ (resp. $\mathcal{N}$), along M_1 and ℓ_5. Furthermore, we can assume that the four added lines are defined by real linear forms, in such way that $\mathfrak{M}^+_{M_1,\ell_5}$ and $\mathfrak{N}^+_{M_1,\ell_5}$ are real-complexified arrangements. For example, we can consider the arrangements $\mathfrak{M}^+_{M_1,\ell_5} = \mathcal{M}\cup\{D_1,D_2,D_3,D_4\}$ and $\mathfrak{N}^+_{M_1,\ell_5} = \mathcal{N}\cup\{D_1,D_2,D_3,D_4\}$, where

$$\begin{array}{ccc}
D_1 : x + y + z = 0 & \text{and} & D_2 : x + y + 2z = 0, \\
D_3 : x + 3y - 5z = 0 & \text{and} & D_4 : x - 3y - 5z = 0.
\end{array}$$

These arrangements are well augmented arrangements of $\mathcal{M}$ and $\mathcal{N}$ since the lines M_1, D_1 and D_2 (resp. ℓ_5, D_3 and D_4) are concurrent; and D_1, D_2, D_3 and D_4 are generic with all the other lines.

Theorem 5.20. *The complexified real arrangements* $\mathfrak{M}^+_{M_1,\ell_5}$ *and* $\mathfrak{N}^+_{M_1,\ell_5}$ *verify the following properties:*

(1) The arrangements $\mathfrak{M}^+_{M_1,\ell_5}$ *and* $\mathfrak{N}^+_{M_1,\ell_5}$ *are combinatorially equivalent.*
(2) There is no homeomorphism of $\mathbb{CP}^2$ *sending* $\mathfrak{M}^+_{M_1,\ell_5}$ *on* $\mathfrak{N}^+_{M_1,\ell_5}$*.*
(3) The complements of $\mathfrak{M}^+_{M_1,\ell_5}$ *and* $\mathfrak{N}^+_{M_1,\ell_5}$ *are homotopy-equivalent.*

Remark 5.21. The homeomorphism type of the arrangements $\mathcal{M}$ and $\mathcal{N}$ are not distinguished by a linking invariant, so we cannot conclude here that the one of $M(\mathfrak{M}^+_{M_1,\ell_5})$ and of $M(\mathfrak{N}^+_{M_1,\ell_5})$ are different.

6. Combinatorial classes of arrangements with connected moduli spaces

One of the difficulties in the study of Zariski pairs is to construct explicit examples of combinatorics with a non-connected moduli space so that two arrangements in distinct connected components may have different topologies, this necessary condition comes from Randell Isotopy Theorem [70]. The purpose of the following two sections will be to study the number of connected components of a moduli space using combinatorial tools.

In this section, we explore combinatorial classes of arrangements with a connected moduli space. The results presented here are in the continuation of the works of: Jiang and Yau [52] where they present the class of *nice* arrangements, Wang and Yau [83] where they introduce the class of *simple* arrangements, and Nazir and Yoshinaga [61] where they define the classes of *inductively connected* and of C_3 *of simple type* arrangements. In particular, we provide a solution to a question of Nazir and Yoshinaga about the relation between all these classes, see [61, Introduction].

6.1. Type of an arrangement & naive dimension.

Let $\mathcal{A} = \{\ell_1, \dots, \ell_n\}$ be an arrangement with order ω. For $i \in \{1, \dots, \}$, we define $\mathcal{A}_i$ the subarrangement of $\mathcal{A}$ defined by $\mathcal{A}_i = \omega^{-1}(\{1, \dots, i\})$. It inherits from $\mathcal{A}$ a structure of order arrangement. Consider the following ascending chain of arrangements:

$$\mathcal{A}_1 = \{\ell_1\} \subsetneq \mathcal{A}_2 \subsetneq \cdots \subsetneq \mathcal{A}_i \subsetneq \cdots \subsetneq \mathcal{A}_n = \mathcal{A}, \tag{AC}$$

We denote by $\tau_i(\mathcal{A}, \omega)$ (alternatively, $\tau(\mathcal{A}, \omega, \ell_i)$ or τ_i depending on the context) the cardinality of the intersection $\#\ell_i \cap \mathrm{Sing}(\mathcal{A}_{i-1})$. Note that we also have:

$$\tau_i = \sum_{P \in \mathcal{C}(\mathcal{A}_i)} (|P| - 2) - \sum_{Q \in \mathcal{C}(\mathcal{A}_{i-1})} (|Q| - 2).$$

Definition 6.1. The *type* of an ordered arrangement $(\mathcal{A}, \omega)$ is the n-tuple defined as

$$\tau(\mathcal{A}, \omega) = (\tau_1, \dots, \tau_n) \in \mathbb{N}^n.$$

The ascending chain (AC) and the type are of combinatorial nature. So they can be defined over an abstract line combinatorics $\mathcal{C}$. By an extension of notation, we will also denote $(\mathcal{C})_1 \subsetneq \cdots \subsetneq (\mathcal{C})_n = \mathcal{C}$ and $\tau(\mathcal{C}, \omega)$.

Example 6.2. Consider the arrangement $\mathcal{A}$ pictured in Figure (10) endowed with the order ω induced by the indices. Its type is $\tau(\mathcal{A}, \omega) = (0, 0, 0, 0, 2, 2, 1, 3)$.

For two ordered arrangements $(\mathcal{A}, \omega)$ and $(\mathcal{A}', \omega')$ of n and n' lines respectively, and such that $\mathcal{A} \cap \mathcal{A}' = \emptyset$ (i.e. without common line, but not necessarily with generic intersection), one can define

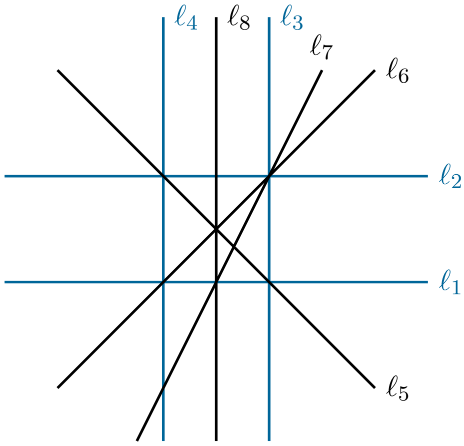


FIGURE 10. An arrangement with type $\tau(\mathcal{A},\omega) = (0,0,0,0,2,2,1,3)$.

the operation $(\mathcal{A} \sqcup \mathcal{A}', \omega \oplus \omega')$ producing an ordered arrangement of $n+n'$ lines with order:

$$\begin{array}{rccl} \omega \oplus \omega' : & \mathcal{A} \sqcup \mathcal{A}' & \longrightarrow & \{1,\dots,n+n'\} \\ & \ell & \longmapsto & \begin{cases} \omega(\ell) & \text{if } \ell \in \mathcal{A}, \\ n+\omega'(\ell) & \text{if } \ell \in \mathcal{A}'. \end{cases} \end{array}$$

Remark 6.3. If $\mathcal{A} \in \mathrm{Arr}_n$ and $\mathcal{A}' \in \mathrm{Arr}_{n'}$ have no common lines, for any orders ω and ω' on $\mathcal{A}$ and $\mathcal{A}'$ respectively, one has that for any $i \in \{1,\dots,n\}$ and $j \in \{1,\dots,n'\}$:

$$\tau_i(\mathcal{A},\omega) = \tau_i(\mathcal{A} \sqcup \mathcal{A}', \omega \oplus \omega') \quad \text{and} \quad \tau_j(\mathcal{A}',\omega') \leq \tau_{n+j}(\mathcal{A} \sqcup \mathcal{A}', \omega \oplus \omega'),$$

Furthermore, for any other order $\overline{\omega}$ on $\mathcal{A}$, and for any $j \in \{1,\dots,n'\}$, one has that:

$$\tau_{n+j}(\mathcal{A} \sqcup \mathcal{A}', \omega \oplus \omega') = \tau_{n+j}(\mathcal{A} \sqcup \mathcal{A}', \overline{\omega} \oplus \omega').$$

Proposition 6.4. *Let $(\mathcal{A},\omega)$ be an ordered arrangement. The following equality holds:*

$$\sum_{i=1}^{n}(2-\tau_i) = 2|\mathcal{A}| - \sum_{P \in \mathcal{C}(\mathcal{A})} (|P|-2).$$

The right-hand side of the equation in the preceding proposition can be seen as a preliminary attempt to establish a combinatorial formula for the dimension of the realization space $\mathcal{R}(\mathcal{A})$, despite its naive nature. The part $2|\mathcal{A}|$ gives the number of required variables in the construction, while the sum over $P \in \mathcal{C}(\mathcal{A})$ corresponds to the restriction imposed by the singular points. Since the action of $\mathrm{PGL}_3(\mathbb{C})$ can fix four points in generic position, this induces a reduction of 8 to pass from the dimension of $\mathcal{R}(\mathcal{A})$ to the dimension of $\mathcal{M}(\mathcal{A})$.

Definition 6.5. Let $\mathcal{A}$ be a line arrangement. The *naive dimension* of the moduli space is:

$$\mathrm{d}^{\mathrm{naive}}\, \mathcal{M}(\mathcal{A}) = 2|\mathcal{A}| - 8 - \sum_{P \in \mathcal{C}(\mathcal{A})} (|P|-2).$$

Remark 6.6. The naive dimension is fully determined by the combinatorics of $\mathcal{A}$, so we can define it on any abstract line combinatorics. Recall that in general the naive dimension is not equal to the dimension of the moduli space, by Pappus' hexagon theorem.

From [28, Remark 4.4], we have the following bound for the dimension of the moduli space.

Proposition 6.7. *Let $\mathcal{A}$ be an arrangement. The following inequality holds:*

$$\mathrm{d}^{\mathrm{naive}}\, \mathcal{M}(\mathcal{A}) \leq \dim_{\mathbb{C}} \mathcal{M}(\mathcal{A}).$$

6.2. **Inductively connected arrangement.**

The results of this section are reformulation and refinement of those of Nazir and Yoshinaga [61]. First, using the notion of the type of an arrangement, we can reformulate the definition of inductively connected arrangement.

Definition 6.8. An arrangement $\mathcal{A}$ is *inductively connected* if there exists an order ω on $\mathcal{A}$ such that

$$\max(\tau(\mathcal{A},\omega)) \leq 2.$$

Example 6.9. The arrangement $\mathcal{A}$ pictured in Figure (11) endowed with the order ω induced by the indices has type $\tau(\mathcal{A},\omega) = (0,0,0,0,2,2,1)$. So, it is inductively connected.

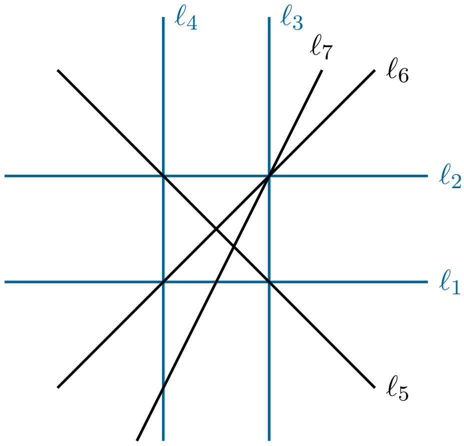


FIGURE 11. An inductively connected arrangement with type $\tau(\mathcal{A},\omega) = (0,0,0,0,2,2,1)$.

In the following, we will consider two particular families of line arrangements. Let n be a positive natural number. We define:

- the family of arrangements formed by a pencil of n lines, denoted by $\mathfrak{X}(n) \subset \mathrm{Arr}_n$,
- the family of arrangements formed by a pencil of $n-1$ lines and a unique generic line, denoted by $\overline{\mathfrak{X}}(n) \subset \mathrm{Arr}_n$.

FIGURE 12. Example of arrangements in the $\mathfrak{X}(4)$ and $\overline{\mathfrak{X}}(4)$.

Proposition 6.10. *If $\mathcal{A}$ is an inductively connected arrangement such that $\mathcal{A} \notin \mathfrak{X}(n) \cup \overline{\mathfrak{X}}(n)$, then $\mathcal{M}(\mathcal{A})$ is isomorphic to a Zariski open subset of a complex space of dimension $\mathrm{d}^{\mathrm{naive}}\mathcal{M}(\mathcal{A})$. As a consequence, the moduli space $\mathcal{M}(\mathcal{A})$ is connected.*

In a larger context, an explicit parametrization of $\mathcal{M}(\mathcal{A})$ as an open Zariski subset of an affine complex space is provided in Theorem 7.4.

Corollary 6.11. *If $\mathcal{A} \in \mathrm{Arr}_n$ is an inductively connected arrangement such that $\mathcal{A} \notin \mathfrak{X}(n) \cup \overline{\mathfrak{X}}(n)$, then*

$$\dim_{\mathbb{C}} \mathcal{M}(\mathcal{A}) = \sum_{i=5}^{n} (2 - \tau_i) = \mathrm{d}^{\mathrm{naive}} \mathcal{M}(\mathcal{A}).$$

Remark 6.12. In the same spirit as [61, Lemma 3.2], let $(\mathcal{A}, \omega)$ and $(\mathcal{A}', \omega')$ be two ordered arrangements such that $\tau_i(\mathcal{A} \sqcup \mathcal{A}', \omega \oplus \omega') \leq 2$ for all $i \in \{|\mathcal{A}| + 1, \dots, |\mathcal{A}| + |\mathcal{A}'|\}$. If $\mathcal{M}(\mathcal{A})$ is irreducible, then $\mathcal{M}(\mathcal{A} \sqcup \mathcal{A}')$ is irreducible thus connected and it has dimension

$$\dim_{\mathbb{C}} \mathcal{M}(\mathcal{A} \sqcup \mathcal{A}') = \dim_{\mathbb{C}} \mathcal{M}(\mathcal{A}) + \sum_{i=|\mathcal{A}|+1}^{|\mathcal{A}|+|\mathcal{A}'|} \big(2 - \tau_i(\mathcal{A} \sqcup \mathcal{A}', \omega \oplus \omega')\big).$$

Note that the reciprocal is not true in general, as it is shown in the following example.

Example 6.13. In Section 4.5, we present a line combinatorics $\mathcal{C}$ of 13 lines with a moduli space with two irreducible components of dimension 1. These components are geometrically characterized, e.g. there are particular multiple points P_1, P_2, and P_3 which are collinear in the first component, but they are not in the second one (see Proposition 4.27). Consider the combinatorics $\mathcal{C}'$ constructed on $\mathcal{C}$ by the addition of a line ℓ_{14} passing through P_1 and P_2 but avoiding P_3. By construction, one has that $\tau_{14}(\mathcal{C}') = 2$ and $\mathcal{M}(\mathcal{C}')$ is irreducible.

6.3. Nice, simple and generalized simple arrangements.

In [52], the Jiang and Yau define the combinatorial class of *nice* arrangements, which is generalized in [83] as the class of *generalized simple* arrangements. In the introduction of [61], Nazir and Yoshinaga ask about the relation between these two combinatorial classes and the class of inductively connected arrangements. The purpose of this section is to prove that any generalized simple arrangement is actually inductively connected. First, let us recall the definition of generalized simple arrangements.

For any arrangement $\mathcal{A}$, we define $\mathcal{G}(\mathcal{A})$ as the graph whose vertices v_P are in one-to-one correspondence with the multiple points P of $\mathcal{A}$, i.e. the singular points whose multiplicity is at least 3; two vertices v_P and v_Q are joined by an edge if and only if the two corresponding points P and Q lie on the same line of $\mathcal{A}$. Associated with this graph, one may define the following notions:

- A *star* centered in v_P, and denoted $\mathrm{St}(v_P)$, is the subgraph of $\mathcal{G}(\mathcal{A})$ generated by v_P and any neighbor vertex of v_P. The open star $\overset{\circ}{\mathrm{St}}(v_P)$ is the complex composed of v_P and any adjacent edge (without the neighbor vertices of v_P).
- A *reduced circle* of $\mathcal{G}(\mathcal{A})$ is a tuple of $(v_1, \dots, v_k)$ of vertices of $\mathcal{G}(\mathcal{A})$, such that for all $i \in \{1, \dots, k\}$, one has that (v_i, v_{i+1}) is an edge of $\mathcal{G}(\mathcal{A})$ and $v_{i-1} \neq v_{i+1}$ (here the indices are considered modulo k).
- A *generalized free net* based on a tuple $(B_0, \dots, B_m)$ of reduced circles of $\mathcal{G}(\mathcal{A})$ (where B_0 can also be a single vertex), is the maximal subgraph $\mathrm{Net}(B_0, \dots, B_m)$ of $\mathcal{G}(\mathcal{A})$ whose vertices are at a distance of at most 1 from at least one of the circles B_i, and such that
  - (N1) for all $i \in \{0, \dots, m\}$, two vertices v and w of B_i are connected by an edge if and only if they are adjacent in B_i,
  - (N2) for all $i, j \in \{0, \dots, m-2\}$ and two vertices $v \in B_i$ and $w \in B_j$, there does not exist $z \in B_k$ with $k > \max(i, j)$ which verifies that (v, z) and (w, z) are both edges in $\mathrm{Net}(B_0, \dots, B_m)$,
  - (N3) for all pair of vertices $v, w \in \mathrm{Net}(B_0, \dots, B_m)$, there does not exist $z \in \mathcal{G}(\mathcal{A}) \setminus \mathrm{Net}(B_0, \dots, B_m)$ which verifies that (v, z) and (w, z) are both edges in $\mathcal{G}(\mathcal{A})$,
  - (N4) for all $i \in \{1, \dots, n\}$, there exists a vertex $v \in B_i$ which is not connected by an edge to any reduced circle B_j with $j < i$.

The open net $\overset{\circ}{\mathrm{Net}}(B_0,\dots,B_m)$ is the complex formed by the vertices of the B_i and all the edges of $\mathrm{Net}(B_0,\dots,B_m)$, i.e. all the end vertices of the net are removed.

Definition 6.14. An arrangement $\mathcal{A}$ is *generalized simple* if there are stars $\mathrm{St}(v_1),\dots,\mathrm{St}(v_k)$ and generalized free nets $\mathrm{Net}_1,\dots,\mathrm{Net}_l$ which are pairwise disjoint in $\mathcal{G}(\mathcal{A})$ and such that the graph

$$\mathcal{G}(\mathcal{A}) \setminus \left\{ \overset{\circ}{\mathrm{St}}(v_1),\dots,\overset{\circ}{\mathrm{St}}(v_k),\overset{\circ}{\mathrm{Net}}_1,\dots,\overset{\circ}{\mathrm{Net}}_l \right\}$$

is a forest. When $l=0$, the arrangement $\mathcal{A}$ is *nice*, and it is *simple* when for all nets have $m=2$.

Theorem 6.15. *If $\mathcal{A}$ is a generalized simple arrangement, then $\mathcal{A}$ is inductively connected.*

Idea of the proof. Using the combinatorial characterization of the generalized simple arrangements, we construct an explicit order such that each τ_i is lesser than 2. The highest lines correspond to those in the forest, then come the one in the stars, and the smallest correspond to the edges in the nets. ☐

6.4. Inductively rigid arrangements.

Following an antagonist approach to the case of inductively connected arrangements, we introduce another combinatorial class of arrangements with connected moduli space.

Definition 6.16. An arrangement $\mathcal{A} \in \mathrm{Arr}_n$ is *inductively rigid* if there exists an order ω such that for any arrangement $\mathcal{A}_i$ in the ascending chain (AC), we have $\mathcal{M}(\mathcal{A}_i) = \{\mathcal{A}_i\}$.

Note that if an arrangement $\mathcal{A}$ is inductively rigid, then any $\mathcal{A}_i$ in the induced ordered chain (AC) is also inductively rigid. As a consequence, any arrangement $\mathcal{A}$ with $|\mathcal{A}| \leq 3$ is inductively rigid. In addition, no arrangement in the classes $\mathfrak{X}(n)$ and $\overline{\mathfrak{X}}(n+1)$ is inductively rigid for $n \geq 4$.

Proposition 6.17. *An arrangement $\mathcal{A} \in \mathrm{Arr}_n$ is inductively rigid if and only if either $n \leq 4$ and $\mathcal{A} \notin \mathfrak{X}(4)$, or $n \geq 5$ and there exists an order ω on $\mathcal{A}$ such that:*

(IR1) $\mathcal{A}_4$ is generic, i.e. for all $i \in \{1,2,3,4\}$, we have that $\tau_i(\mathcal{A},\omega)=0$,
(IR2) for all $i \in \{5,\dots,n\}$, we have that $\tau_i(\mathcal{A},\omega) \geq 2$.

As an important consequence of the previous proposition, we have that the class of inductively rigid arrangements is combinatorial.

Example 6.18. Consider the arrangement $\mathcal{B}$ given in Figure (13), with order ω induced by the line numbering. It is inductively rigid since its type is $\tau(\mathcal{B},\omega) = (0,0,0,0,2,2,2,2,2,3)$. However, it is not inductively connected since all the lines contain at least 3 multiple points. Furthermore, it is not C_3, so it is neither C_3 of simple type (see Section 6.5 for the definitions).

6.5. Arrangements with a rigid pencil form.

An arrangement $\mathcal{A}$ is C_k if k is the minimal integer such that there exists a subarrangement $\mathcal{D}_k \subset \mathcal{A}$ of k lines such that $\mathrm{Sing}_{\geq 3}(\mathcal{A})$ is contained in $\mathcal{D}_k$. Nazir and Yoshinaga proved that if an arrangement is C_0, C_1 or C_2 then it is inductively connected, and so that its moduli space is connected, see [61, Theorem. 3.11]. They also introduced the combinatorial class C_3 of simple[18] type.

Definition 6.19. An arrangement $\mathcal{A}$ of class C_3 is of *simple type* if one of the following conditions holds:

(i) the three lines of $\mathcal{D}_3$ are in generic position, and one of them contains a unique multiple point,
(ii) the three lines of $\mathcal{D}_3$ are concurrent.

[18]The term simple used here is not related to the one of Section 6.3.

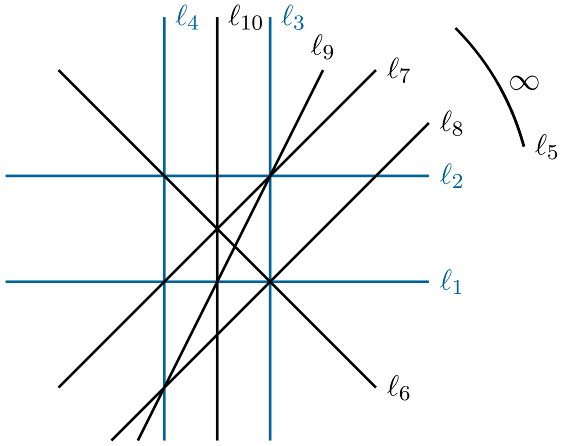


FIGURE 13. An inductively rigid arrangement which is neither inductively connected, nor C_3 of simple type.

Theorem 6.20 ([61, Theorem. 3.15])**.** *If $\mathcal{A}$ is a C_3 arrangement of simple type, then its moduli space $\mathcal{M}(\mathcal{A})$ is connected.*

We can generalize this class of arrangements, thereby freeing them from the constraint of being C_3.

Definition 6.21. An arrangement $\mathcal{A}$ has a *rigid pencil form* if it contains an inductively rigid subarrangement $\mathcal{A}'$ with a singular point P_0, such that for any multiple point $P \in \mathrm{Sing}_{\geq 3}(\mathcal{A})$, one of the following holds:

(1) P is a singular point of $\mathcal{A}'$,
(2) the line (P, P_0) is in $\mathcal{A}'$.

Proposition 6.22. *Any C_3 arrangement of simple type has a rigid pencil form.*

Theorem 6.23. *If $\mathcal{A}$ has a rigid pencil form, then its moduli space $\mathcal{M}(\mathcal{A})$ is connected.*

Idea of the proof. If $\mathcal{A}$ has a rigid pencil form then all its singular points are contained in a fixed pencil or are in $\mathcal{A}'$. This allows to express the moduli space as an open Zariski subset of the kernel of an affine application with complex coefficients, as done in [61, Theorem 3.15]. As a consequence, the moduli space is irreducible and so connected. $\square$

Example 6.24. Consider the arrangement $\mathcal{A} \in \mathrm{Arr}_{10}$ pictured in Figure (14) and defined by the equations:

$$\begin{array}{cccc} \ell_1 : x = 0, & \ell_2 : x - z = 0, & \ell_3 : y = 0, & \ell_4 : y - z = 0, \\ \ell_5 : z = 0, & \ell_6 : -x + y = 0, & \ell_7 : x + y - z = 0, & \ell_8 : -2x + 4y - z = 0, \\ \ell_9 : 2x - 3y + z = 0, & \ell_{10} : -4x + 6y = 0. & & \end{array}$$

Its combinatorics is given by:

$$\left\{ \begin{array}{c} \{\ell_1, \ell_2, \ell_5\}, \{\ell_1, \ell_3, \ell_6, \ell_{10}\}, \{\ell_1, \ell_4, \ell_7\}, \{\ell_1, \ell_8\}, \{\ell_1, \ell_9\}, \{\ell_2, \ell_3, \ell_7\} \\ \{\ell_2, \ell_4, \ell_6, \ell_9\}, \{\ell_2, \ell_8\}, \{\ell_2, \ell_{10}\}, \{\ell_3, \ell_4, \ell_5\}, \{\ell_3, \ell_8, \ell_9\}, \{\ell_4, \ell_8, \ell_{10}\} \\ \{\ell_5, \ell_6\}, \{\ell_5, \ell_7\}, \{\ell_5, \ell_8\}, \{\ell_5, \ell_9, \ell_{10}\}, \{\ell_6, \ell_7, \ell_8\}, \{\ell_7, \ell_9\}, \{\ell_7, \ell_{10}\} \end{array} \right\}.$$

Let $\mathcal{A}' = \{\ell_1, \cdots \ell_7\}$ be the subarrangement of $\mathcal{A}$ with combinatorics:

$$\{\{\ell_1, \ell_2, \ell_5\}, \{\ell_1, \ell_3, \ell_6\}, \{\ell_1, \ell_4, \ell_7\}, \{\ell_2, \ell_3, \ell_7\}, \{\ell_2, \ell_4, \ell_6\}, \{\ell_3, \ell_4, \ell_5\}, \{\ell_5, \ell_6\}, \{\ell_5, \ell_7\}, \{\ell_6, \ell_7\}\}.$$

Once again, if we consider the order ω induced by the indices, we have $\tau(\mathcal{A}', \omega) = (0, 0, 0, 0, 2, 2, 2)$ and thus $\mathcal{A}'$ is an inductively rigid arrangement. Furthermore, all multiple points of $\mathcal{A}$ are contained in the three concurrent lines ℓ_3, ℓ_4 and ℓ_5, except for the point $\{\ell_6, \ell_7, \ell_8\}$ which contains the two lines ℓ_6 and ℓ_7 of $\mathcal{A}'$. Thus, $\mathcal{A}$ has a rigid pencil form. By Theorem 6.23, we deduce that $\mathcal{M}(\mathcal{A})$ is connected.

It is worth noticing that all multiple points of $\mathcal{A}$ are also contained in the 4-pencil $\{\ell_1, \ell_3, \ell_6, \ell_{10}\}$ or $\{\ell_2, \ell_4, \ell_6, \ell_9\}$. This implies that $\mathcal{A}$ is a C_4 arrangement. Nevertheless, any of these two 4-pencils is contained in an inductively rigid subarrangement of $\mathcal{A}$. So $\mathcal{A}$ does not admit a rigid pencil form.

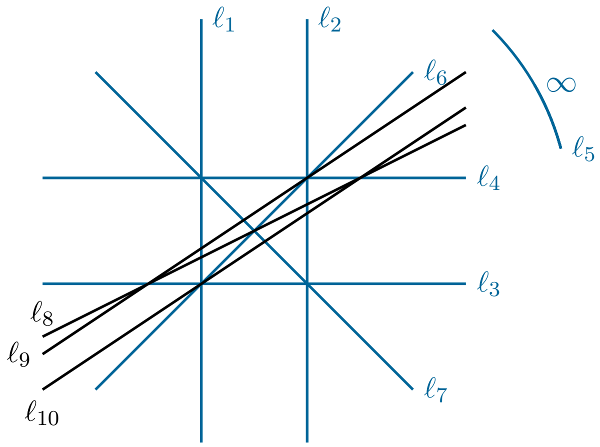


FIGURE 14. An arrangement with rigid pencil form.

Remark 6.25. The statement of Theorem 6.23 still holds if the subarrangement $\mathcal{A}'$ verifies that $\mathcal{M}(\mathcal{A}') = \{\mathcal{A}'\}$ instead of being inductively rigid. Nevertheless, one should notice that this is a geometric but not combinatorial condition.

7. On the number of connected components of $\mathcal{M}(\mathcal{A})$

We know since MacLane [56] that the moduli space of a line arrangement can be non-connected, nevertheless, the behavior of the number of connected components has never been studied. The objective of this section is to introduce a strategy to inductively build an upper bound on the number of connected components of $\mathcal{M}(\mathcal{A})$. The foundation of this method is the class of *inductively connected* arrangements introduced by Nazir and Yoshinaga [61], see also Definition 6.8.

7.1. **Perturbations of a combinatorics.**

Let $\mathcal{A}$ be an arrangement and P be a multiple point of $\mathcal{Z}(\mathcal{A})$. Assume that $\ell \in \mathcal{A}$ passes through P, a small perturbation of ℓ near P modifies the combinatorics of $\mathcal{A}$ and transforms P into a singular point $\widetilde{P}$ of multiplicity $m(P) - 1$. The following definition mimics this geometric perturbation at a combinatorial level.

Definition 7.1. Let $\mathcal{C} = (\mathcal{L}, \mathcal{P})$ be an abstract line combinatorics. Let $P_0 \in \mathcal{P}$ be a fixed multiple point of $\mathcal{C}$ and let $\ell \in P_0$. An *elementary perturbation* of $\mathcal{C}$ at (ℓ, P_0) is an abstract combinatorics $\widetilde{\mathcal{C}} = (\mathcal{L}, \widetilde{\mathcal{P}})$ such that:

(1) for all $P \in \mathcal{P}$, $P \neq P_0$, one has $P \in \widetilde{\mathcal{P}}$,
(2) $\widetilde{P}_0 = P_0 \setminus \{\ell\}$ is in $\widetilde{\mathcal{P}}$.

Such a relation is denoted $\widetilde{\mathcal{C}} \prec \mathcal{C}$. For any multiple point $P \in \mathcal{P}$, we call the *parent of* P in $\widetilde{\mathcal{C}}$ the unique element $\widetilde{P} \in \widetilde{\mathcal{P}}$ such that either $\widetilde{P} = P$ or $\widetilde{P} = \widetilde{P}_0$.

Example 7.2. Let $\mathcal{C} = \{\{\ell_1,\ell_2,\ell_3,\ell_4\},\{\ell_1,\ell_5\},\{\ell_2,\ell_5\},\{\ell_3,\ell_5\},\{\ell_4,\ell_5\}\}$ be the combinatorics of an arrangement in $\overline{\mathfrak{X}}(5)$. The elementary perturbation of $\mathcal{C}$ at (P_0,ℓ_4), for $P_0 = \{\ell_1,\ell_2,\ell_3,\ell_4\}$, is:

$$\{\{\ell_1,\ell_2,\ell_3\},\{\ell_1,\ell_4\},\{\ell_1,\ell_5\},\{\ell_2,\ell_4\},\{\ell_2,\ell_5\},\{\ell_3,\ell_5\},\{\ell_4,\ell_5\},\{\ell_4,\ell_5\}\},$$

and the parent of $P_0 = \{\ell_1,\ell_2,\ell_3,\ell_4\}$ is $\{\ell_1,\ell_2,\ell_3\}$.

Definition 7.3. A *m-perturbation* of $\mathcal{C} = (\mathcal{L},\mathcal{P})$ is a sequence of m elementary perturbations such that

$$\mathcal{C}_0 \prec \mathcal{C}_1 \prec \cdots \prec \mathcal{C}_m = \mathcal{C},$$

and $\mathcal{C}_0$ is inductively connected with associated order ω_0. Since $\mathcal{C}_0$ and $\mathcal{C}$ have the same set of lines, then ω_0 is also an order on $\mathcal{C}$, and it will be called the *perturbation order*. For any multiple point P in $\mathcal{P}_i$, there is a unique chain of parents coming from each elementary perturbation:

$$\widetilde{P} = P_0 \subset P_1 \subset \cdots \subset P_i = P.$$

The point $\widetilde{P}$ in $\mathcal{C}_0$ is called the *ancestor of P*.

For brevity's sake, an m-perturbation $\mathcal{C}_0 \prec \mathcal{C}_1 \prec \cdots \prec \mathcal{C}_m = \mathcal{C}$ will be denoted by $\mathcal{C}_0 \prec\!\!\prec \mathcal{C}$. Since the generic combinatorics is inductively connected, and since it can be obtained from any $\mathcal{C}$ by a sequence of elementary perturbations, then any combinatorics $\mathcal{C}$ admits a m-perturbation for some m.

7.2. Parametrization of the moduli space.

The purpose of this section is to give a constructible proof of the following result.

Theorem 7.4. *Let $\mathcal{A} \in \mathrm{Arr}_n$ be an arrangement, with $\mathcal{A} \notin \mathfrak{X}(n) \cup \overline{\mathfrak{X}}(n)$, which admits a m-perturbation $\mathcal{C}_0 \prec\!\!\prec \mathcal{C}(\mathcal{A})$, and denote $d_0 = \mathrm{d}^{\mathrm{naive}}\,\mathcal{M}(\mathcal{C}_0)$. There exists an open Zariski subset W_0 of the affine space of dimension d_0 and m polynomials $\Delta_1,\dots,\Delta_m$ such that*

$$\mathcal{M}(\mathcal{A}) \simeq \{(v_1,\dots,v_{d_0}) \in W_0 \mid \Delta_1 = \cdots = \Delta_m = 0\}.$$

Remark that we may have that some of the Δ_i's are always trivial. It is, for example, the case in Pappus arrangement. From the previous result, we can deduce the following corollary. The left-hand inequality corresponds to the one of Proposition 6.7.

Corollary 7.5. *Let $\mathcal{A} \in \mathrm{Arr}_n$ be a line arrangement, with $\mathcal{A} \notin \mathfrak{X}(n) \cup \overline{\mathfrak{X}}(n)$. If $\mathcal{C}(\mathcal{A})$ admits a m-perturbation, then*

$$0 \leq \dim_{\mathbb{C}} \mathcal{M}(\mathcal{A}) - \mathrm{d}^{\mathrm{naive}}\,\mathcal{M}(\mathcal{A}) \leq m.$$

The strategy of the proof of Theorem 7.4 is the following. The goal is to express the moduli space $\mathcal{M}(\mathcal{A})$ from the following data:

- An open Zariski subset W_0 of an affine linear space, codifying the open conditions.
- A map Ψ over W_0 which parametrizes the arrangements in Arr_n verifying the Zariski-closed conditions of $\mathcal{M}(\mathcal{C}_0)$, e.g those of type $\Delta_{i,j,k} = 0$ with $\{\ell_i,\ell_j,\ell_k\} \subset \widetilde{P}$ for a multiple point $\widetilde{P}$ in $\mathcal{C}_0$.
- A list $\Delta_1,\dots,\Delta_m$ of m polynomials, i.e. the remaining Zariski-closed conditions coming from multiple points P in $\mathcal{C}(\mathcal{A})$ which are not in $\mathcal{C}_0$, where each polynomial represents a step of the m-perturbation $C_0 \prec\!\!\prec \mathcal{C}$.

We start with the construction of an expression of the map Ψ, then we give the polynomials $\Delta_1,\dots,\Delta_m$ and finally the existence of W_0 is discussed.

Remark 7.6. Let $P = \{\ell_{i_1}, \dots, \ell_{i_m}\}$ be a multiple point in $\mathcal{C}(\mathcal{A})$. One can reduce the set of Zariski-closed conditions $\{\Delta_{i,j,k} = 0 \mid \{\ell_i, \ell_j, \ell_k\} \subset P\}$ to its subset $\{\Delta_{i_1,i_2,i_k} = 0 \mid k = 3, \dots, m\}$.

Let $\mathbf{v} = (v_1, \dots, v_{d_0})$ be a system of coordinates in $V_0 = \mathbb{C}^{d_0}$. For each line ℓ_i, we associate three polynomials $a_i, b_i, c_i \in \mathbb{C}[v_1, \dots, v_{d_0}]$ such that

$$\ell_i : a_i(\mathbf{v})x + b_i(\mathbf{v})y + c_i(\mathbf{v})z = 0.$$

In this way, one can define a map called *parametrization* of $\mathcal{M}(\mathcal{A})$:

$$\Psi : \mathbf{v} \in W_0 \longmapsto \left(\begin{pmatrix} a_1(\mathbf{v}) \\ b_1(\mathbf{v}) \\ c_1(\mathbf{v}) \end{pmatrix}, \dots, \begin{pmatrix} a_n(\mathbf{v}) \\ b_n(\mathbf{v}) \\ c_n(\mathbf{v}) \end{pmatrix} \right) \in (\mathbb{C}^3)^n.$$

In addition to Ψ, one can construct another map Φ that parametrize the singular points of the elements in $\mathcal{M}(\mathcal{A})$ as follows. Let P be a multiple point in the combinatorics $\mathcal{C}(\mathcal{A})$, and let $\widetilde{P}$ be the ancestor of P in $\mathcal{C}_0$. Take ℓ_i and ℓ_j the two lines in $\widetilde{P}$ which are minimal with respect to the order ω_0. We define

$$\Phi_P = (b_i c_j - b_j c_i, a_j c_i - a_i c_j, a_i b_j - a_j b_i) \in \mathbb{C}[v_1, \dots, v_{d_0}]^3,$$

If an arrangement $\mathcal{A}_0 \in \mathcal{M}(\mathcal{A})$ is given by $\Psi(\mathbf{v})$ then for any multiple point P of $\mathcal{C}(\mathcal{A})$, the vector $\Phi_P(\mathbf{v}) \in \mathbb{C}^3$ express the homogeneous coordinates of P in $\mathbb{CP}^2$.

Let us describe in detail how the polynomials a_i, b_i and c_i are inductively constructed. Since $\mathcal{A} \notin \mathfrak{X}(n) \cup \overline{\mathfrak{X}}(n)$, one can assume that the perturbation order ω_0 is such that the lines ℓ_1, ℓ_2, ℓ_3 and ℓ_4 are in generic position in both $\mathcal{C}_0$ and $\mathcal{C}(\mathcal{A})$. Using the action of $\mathrm{PGL}_3(\mathbb{C})$, we fix them as $x = 0$, $x - z = 0$, $y = 0$ and $y - z = 0$, respectively. In other words, for all $\mathbf{v} \in V_0$, we define

$$\Psi(\mathbf{v})_1 = (1, 0, 0), \quad \Psi(\mathbf{v})_2 = (1, 0, -1), \quad \Psi(\mathbf{v})_3 = (0, 1, 0), \quad \Psi(\mathbf{v})_4 = (0, 1, -1).$$

It follows from its definition that the parametrization Φ of the singular points of these four lines are given for any $\mathbf{v} \in V_0$ by:

$$\begin{array}{lll} \Phi_{\langle \ell_1, \ell_2 \rangle}(\mathbf{v}) = (0, 1, 0), & \Phi_{\langle \ell_1, \ell_3 \rangle}(\mathbf{v}) = (0, 0, 1), & \Phi_{\langle \ell_1, \ell_4 \rangle}(\mathbf{v}) = (0, 1, 1), \\ \Phi_{\langle \ell_2, \ell_3 \rangle}(\mathbf{v}) = (1, 0, 1), & \Phi_{\langle \ell_2, \ell_4 \rangle}(\mathbf{v}) = (1, 1, 1), & \Phi_{\langle \ell_3, \ell_4 \rangle}(\mathbf{v}) = (1, 0, 0). \end{array}$$

where $\langle \ell_{i_1}, \dots, \ell_{i_k} \rangle$ with $k \geq 2$ is the unique point P in $\mathcal{C}(\mathcal{A})$ verifying that $\{\ell_{i_1}, \dots, \ell_{i_k}\} \subset P$, if it exists.

The induction process goes as follows. Assume that the maps Ψ and Φ parametrize $\mathcal{A}_{i-1}$ in the chain (AC) with respect to the perturbation order ω_0. Note that the number of parameters used in this parametrization is $d_i = \sum_{j=5}^{i-1} \left(2 - \tau_j(\mathcal{C}_0, \omega_0)\right)$. The next step is then to extend these parametrization maps to $\mathcal{A}_i$. They are determined by the values $\tau_i(\mathcal{C}_0, \omega_0)$:

- If $\tau_i(\mathcal{C}_0, \omega_0) = 0$, i.e. the line ℓ_i is generic in $(\mathcal{C}_0)_i$. Since $\ell_i \notin \langle \ell_1, \ell_2 \rangle$ or $\ell_i \notin \langle \ell_1, \ell_3 \rangle$ in $\mathcal{C}(\mathcal{A})$, we can parametrize ℓ_i using only two complex parameters v_{d_i} and v_{d_i+1},

$$\Psi(\mathbf{v})_i = \begin{cases} (v_{d_i}, 1, v_{d_i+1}) & \text{if } \ell_i \notin \langle \ell_1, \ell_2 \rangle, \\ (1, v_{d_i}, v_{d_i+1}) & \text{if } \ell_i \notin \langle \ell_1, \ell_3 \rangle. \end{cases}$$

- If $\tau_i(\mathcal{C}_0, \omega_0) = 1$, i.e. the line ℓ_i passes through a unique multiple point P_0 in $(\mathcal{C}_0)_i$. Take ℓ_j and ℓ_k the two lines in P_0 which are minimal with respect to the order ω_0. One defines:

$$\Psi(\mathbf{v})_i = \Psi(\mathbf{v})_j + v_{d_i} \cdot \Psi(\mathbf{v})_k.$$

- If $\tau_i(\mathcal{C}_0,\omega_0)=2$, i.e. the line ℓ_i passes through two multiple points P_0 and Q_0 in $(\mathcal{C}_0)_i$. The line ℓ_i is then parametrized by

$$\Psi(\mathbf{v})_i = \Phi_{\langle P_0\rangle}(\mathbf{v}) \times \Phi_{\langle Q_0\rangle}(\mathbf{v}),$$

where "$\times$" stands for the usual cross product.

The next step is to construct the polynomials $\Delta_1,\dots,\Delta_m$. Assume that the ith elementary perturbation $\mathcal{C}_{i-1} \prec \mathcal{C}_i$ in $\mathcal{C}_0 \lll \mathcal{C}$ is at (ℓ_j, P) and let $\widetilde{P}$ be the parent of P in $\mathcal{C}_{i-1}$. Take two distinct elements $\ell_{i_1},\ell_{i_2} \in \widetilde{P}$ and define:

$$\Delta_i(\mathbf{v}) = \det\left(\Psi(\mathbf{v})_j, \Psi(\mathbf{v})_{i_1}, \Psi(\mathbf{v})_{i_2}\right) \in \mathbb{C}[v_1,\dots,v_{d_0}]. \tag{D}$$

It is worth noticing that another choice of $\ell_{i_1},\ell_{i_2} \in \widetilde{P}$ will lead to an equivalent construction of $\mathcal{M}(\mathcal{A})$, due to Remark 7.6.

The Zariski open subset W_0 can be expressed similarly as

$$W_0 = \{\mathbf{v}\in\mathbb{C}^{d_0} \mid \det\left(\Psi(\mathbf{v})_i,\Psi(\mathbf{v})_j,\Psi(\mathbf{v})_k\right)\neq 0, \text{ if } \nexists P \text{ in } \mathcal{C}(\mathcal{A}) \text{ such that } \{\ell_i,\ell_j,\ell_k\}\subset P\}.$$

By hypothesis $\mathcal{A}\notin\mathfrak{X}(n)\cup\overline{\mathfrak{X}}(n)$, so the above polynomial inequalities imply that $\ell_i\neq\ell_j$, for any $i\neq j\in\{1,\dots,n\}$.

7.3. **Inductive upper bound.**

Using the description of the moduli space obtained in Theorem 7.4, we construct an upper bound of the number of irreducible components of the moduli space. Since $\#\,\mathrm{CC}(\mathcal{M}(\mathcal{A})) \leq \#\,\mathrm{Irr}(\mathcal{M}(\mathcal{A}))$, e.g. [74, Sec. 7.2], this will also provide an upper-bound on the number of connected components. When $\mathcal{A}$ is in $\mathfrak{X}(n)$ or $\overline{\mathfrak{X}}(n)$, we know that its moduli space $\mathcal{M}(\mathcal{A})$ is irreducible and so connected. Among this section, we assume that $\mathcal{A}\notin\mathfrak{X}(n)\cup\overline{\mathfrak{X}}(n)$.

Let $\mathcal{A}=\{\ell_1,\dots,\ell_n\}$ be an arrangement with a m-perturbation $\mathcal{C}_0 \lll \mathcal{C}(\mathcal{A})=(\mathcal{A},\mathcal{P})$ and perturbation order ω_0. Up to relabelling, we assume that $\omega_0(\ell_i)=i$, i.e. ω_0 is the order induced by the indices. Furthermore, we can also assume that ℓ_1, ℓ_2, ℓ_3 and ℓ_4 are in generic position in $\mathcal{C}(\mathcal{A})$. We define recursively two applications $\Lambda:\mathcal{A}\to\mathbb{N}$ and $\Theta:\mathcal{P}\to\mathbb{N}$ as follows.

(R1) For $i\in\{1,2,3,4\}$, we fix $\Lambda(\ell_i)=0$.

(R2) For any $P\in\mathcal{C}(\mathcal{A}_4)$, we fix $\Theta(\langle P\rangle)=0$.

(R3) For $i\in\{5,\dots,n\}$, following the ideas of the proof in Section 7.2, the expression of $\Lambda(\ell_i)$ depends on the value $\tau_i(\mathcal{C}_0,\omega_0)$.

- If $\tau_i(\mathcal{C}_0,\omega_0)=0$, we fix

$$\Lambda(\ell_i)=1.$$

- If $\tau_i(\mathcal{C}_0,\omega_0)=1$, i.e. the line ℓ_i passes through a unique singular point P_0 in $(\mathcal{C}_0)_i$, we define:

$$\Lambda(\ell_i)=\begin{cases} 1+\Lambda(\ell_j) & \text{if } \Lambda(\ell_j)=\Lambda(\ell_k),\\ \max(\Lambda(\ell_j),\Lambda(\ell_k)) & \text{otherwise.}\end{cases}$$

where ℓ_j and ℓ_k are two lines in the parent of P_0.

- If $\tau_i(\mathcal{C}_0,\omega_0)=2$, i.e. the line ℓ_i passes through two singular points P_0 and Q_0 in $(\mathcal{C}_0)_i$, we define

$$\Lambda(\ell_i)=\Theta(\langle P_0\rangle)+\Theta(\langle Q_0\rangle).$$

(R4) For $i\in\{5,\dots,n\}$ and for any double point $P=\{\ell_i,\ell_j\}$ in $(\mathcal{C}_0)_i$, we define

$$\Theta(\langle P\rangle)=\Lambda(\ell_i)+\Lambda(\ell_j).$$

Theorem 7.7. *Let $\mathcal{A}$ be a line arrangement and assume that $\mathcal{C}(\mathcal{A})$ admits a m-perturbation $\mathcal{C}_0 \prec\!\!\prec \mathcal{C}(\mathcal{A})$. The following inequality holds:*

$$\#\,\mathrm{CC}(\mathcal{M}(\mathcal{A})) \leq \prod_{i=1}^{m} \big(\Lambda(\ell_j) + \Lambda(\ell_{i_1}) + \Lambda(\ell_{i_2})\big).$$

where $\mathcal{C}_{i-1} \prec \mathcal{C}_i$ is the elementary perturbation at (ℓ_j, P), and ℓ_{j_1}, ℓ_{j_2} are two lines contained in the parent of P in $\mathcal{C}_{i-1}$.

Proof. In Theorem 7.4, we obtain a description of $\mathcal{M}(\mathcal{A})$ as the intersection of at most m proper algebraic hypersurfaces $V_i : \Delta_i = 0$, defined in Equation (D). By [40, Ex. 8.3.6], this implies that $\#\,\mathrm{Irr}(\mathcal{M}(\mathcal{A})) \leq \prod_{i=1}^{m} \deg \Delta_i$.

Furthermore, the description of Λ and Θ given above ensures that:

$$\deg \Delta_i \leq \Lambda(\ell_j) + \Lambda(\ell_{i_1}) + \Lambda(\ell_{i_2}).$$

□

7.4. **Sharpness of the upper bound.**

By construction, the inequality in Theorem 7.7 becomes an equality for any inductively connected or inductively rigid arrangement. So, the question is: "is this inequality still sharp in non-trivial cases?".

Let p be a prime number such that $p \geq N$. Fix a primitive p-root of unity ζ. Consider the arrangement $\mathcal{A}_p = \{\ell_1, \dots, \ell_{2p+2}\}$ with lines:

$$\ell_1 \colon x - y = 0, \quad \ell_2 \colon x - \zeta y = 0, \quad \ell_{2i+1} \colon x + \zeta^i z = 0 \quad \text{and} \quad \ell_{2i+2} : \zeta^{-i} y + z = 0,$$

for $i \in \{1, \dots, p\}$. The multiple points of $\mathcal{A}_p$ are the two points of multiplicity p:

$$\{\ell_3, \ell_5, \dots, \ell_{2p+1}\} \quad \text{and} \quad \{\ell_4, \ell_6, \dots, \ell_{2p+2}\},$$

and the $2p$ triple points given by:

$$\{\ell_1, \ell_{2i+1}, \ell_{2i+2}\} \quad \text{and} \quad \{\ell_2, \ell_{2i+2}, \ell_{[2i]+3}\}$$

for $i \in \{1, \dots, p\}$, where $[a]$ is the value of a modulo $2p$ such that $0 \leq [a] < 2p$.

Remark 7.8. The arrangement $\mathcal{A}_2$ corresponds to the Ceva arrangement, while the arrangements $\mathcal{A}_3$ are the MacLane arrangements [56]. This family of arrangements appears in [18] in the context of Zariski pairs as subarrangements of the reflection arrangements associated with $G(N, N, 3)$.

Consider the elementary perturbation $\widetilde{\mathcal{C}}(\mathcal{A}_p)$ on $\mathcal{C}(\mathcal{A}_p)$ at $(\ell_{2p+2}, \{\ell_2, \ell_3, \ell_{2p+2}\})$. Let ω_0 be the following order on $\widetilde{\mathcal{C}}(\mathcal{A}_p)$:

$$(\ell_1, \dots, \ell_{2p+2}) \longmapsto (5, 6, 1, 2, 3, 4, 7, 8, \dots, (2p+2)).$$

We have that $\tau(\widetilde{\mathcal{C}}(\mathcal{A}_p), \omega_0) = (0, 0, 0, 0, 2, 1, 2, 2 \dots, 2, 2)$. So $\widetilde{\mathcal{C}}(\mathcal{A}_p)$ is inductively connected, and thus it is a 1-perturbation of $\mathcal{C}(\mathcal{A}_p)$. Using the rules (R1)-(R2)-(R3)-(R4), we can compute that:

$$\Lambda_p : (\ell_1, \dots, \ell_{2p+2}) \longmapsto (0, 1, 0, 0, 0, 0, 1, 1, 2, 2, \dots, p-2, p-2).$$

By Theorem 7.7, we obtain that $\mathcal{M}(\mathcal{A}_p)$ has at most $\Lambda_p(\ell_2) + \Lambda_p(\ell_3) + \Lambda_p(\ell_{2p+2}) = 1 + 0 + (p-2) = p-1$ connected components. On the other hand, the computation of the moduli space using Theorem 7.4 shows that the dimension is zero. Furthermore, it contains $\mathcal{A}_p$ for any choice of primitive p-root of unity ζ. It follows that $\#\,\mathrm{CC}(\mathcal{M}(\mathcal{A}_p)) \geq p - 1$. We can thus state the following theorem.

Theorem 7.9. *For any $N \in \mathbb{N}_{\geq 2}$, there exists an arrangement $\mathcal{A}$ such that $\#\operatorname{CC}(\mathcal{M}(\mathcal{A})) \geq N$, and*

$$\#\operatorname{CC}(\mathcal{M}(\mathcal{A})) = \prod_{i=1}^{m} \left(\Lambda(\ell_j) + \Lambda(\ell_{i_1}) + \Lambda(\ell_{i_2})\right),$$

with the notation of Theorem 7.7.

8. Perspectives for future research

In this section, we discuss topics and problems that will be studied in the future. We have answered some old questions by Falk and Randell [36], or Suciu [78], or Nazir and Yoshinaga [61], but new ones emerged from our works. Let us take a look at some of them and see what they are all about.

8.1. Topology of line arrangements.

In their survey papers, Falk and Randell [35, 36] addressed numerous interesting questions about the homotopy of hyperplane arrangements. Here are some questions that arise from, or that are related to, our work.

8.1.1. *Homotopy vs Fundamental group.*

In [43], we prove the existence and exhibit π_1-equivalent and homotopy-equivalent Zariski pairs. Nevertheless, we do not know if the π_1-equivalent Zariski pairs are also homotopy-equivalent. So, the following question naturally appears.

Question 8.1. *If two combinatorially-equivalent arrangements are π_1-equivalent, then are they necessarily homotopy-equivalent?*

In the previous question, is the answer the same when we drop the combinatorially-equivalent hypothesis? To solve this question will be a natural pursuit of this former result and will answer an old question of Falk [32] questioning the homotopy type of generic sections of the parallel connection of two arrangements.

8.1.2. *Topology vs Complement.*

In the literature, the definition of the topology of an arrangement differs from one article to another. Sometimes it is the homeomorphism type of the pair $(\mathbb{CP}^2, \mathcal{Z}(\mathcal{A}))$ (as we choose in this manuscript), at other times it is the homeomorphism type of the complement $M(\mathcal{A})$.

Question 8.2. *Is the homeomorphism type of $(\mathbb{CP}^2, \mathcal{Z}(\mathcal{A}))$ determined by the one of $M(\mathcal{A})$?*

To positively answer to this question, we should reconstruct the pair $(\mathbb{CP}^2, \mathcal{Z}(\mathcal{A}))$ from the complement. Such an attempt could lead either to an obstruction in the construction and then to a difference, then it would be interesting to construct an explicit example where these two notions differ, or we will succeed in reconstructing the pair $(\mathbb{CP}^2, \mathcal{Z}(\mathcal{A}))$ and then we will obtain an equivalence between these definitions. It is worth noticing that this equivalence is not true for algebraic plane curves in general, see [9].

8.1.3. *Arrangements of 10 lines.*

In [86], Ye proves the classification of the moduli space of 9 lines arrangements suggested by Nazir and Yoshinaga in [61]. A consequence of this classification is the combinatorial determination of the topology for any arrangement with at most 9 lines. On the opposite, Artal, Carmona, Cogolludo and Marco gave an example of a Zariski pair with 11 lines, see also [45] for another example with non-isomorphic fundamental groups.

Problem 8.3. *Classify the topology of* 10 *line arrangements.*

A detailed list of potential Zariski pairs of 10 lines is known [2, 1, 22]. Among this list, I would suggest focusing on complexified real arrangements with a trivial group of automorphism, so that any invariant of the ordered and oriented topology becomes a topological invariant. Then, an invariant based on the braid monodromy and adapted to these specific cases, in the spirit of [7], could allow us to distinguish their topologies. Furthermore, the increasing computation capacity of the modern computer could help us to solve this long-standing question.

8.1.4. *Characteristic varieties.*

The characteristic varieties of a line arrangement are defined as the jumping loci of the homology with local coefficients of the complement. More precisely, the ith characteristic variety is:

$$V_i(\mathcal{A}) = \{\xi \in (\mathbb{C}^*)^n \mid \dim_{\mathbb{C}} \mathrm{H}^1(M(\mathcal{A}); \mathbb{C}_\xi) \geq i\}.$$

The description of their structure started with the work of Arapura [3] where he proved that the irreducible components of $V_i(\mathcal{A})$ are subtori translated by torsion elements[19].

Question 8.4. *Are the characteristic varieties $V_i(\mathcal{A})$ determined by the combinatorics of $\mathcal{A}$?*

By the work of Artal [6], see also [55], we know that the depth[20] of a character ξ depends on the value of the $\mathcal{I}$-invariant of the inner-cyclic triples of the form $(\mathcal{A}, \xi, \gamma)$. Unpublished computations show that the depth of a character ξ in a unique triangular inner-cyclic triple is determined by the combinatorics. It could be interesting to pursue this investigation of the relation between the characteristic varieties and the $\mathcal{I}$-invariant.

8.2. **Linking invariants.**

The linking invariants are recent invariants in the study of the topology of line arrangements. The first definition of the $\mathcal{I}$-invariant appeared in 2013. Nevertheless, during this last decade, they proved many times their efficiency [41, 48, 18, 45, 44]. So they deserve a more in-depth investigation.

8.2.1. *Refinement of the loop-linking numbers.*

Currently, Rodau, a Ph.D. student of Artal and Florens, is constructing a finer version of the loop-linking number. During a private discussion, he explained the construction, and we succeeded in detecting new examples of Zariski pairs. This shows that this invariant is getting closer to the optimal version of a linking invariant. So all the work made in [45] has to be generalized for this new invariant. We could also question its behavior in the case of complexified real arrangements. Is there a diagrammatic version of this invariant in the spirit of the chamber weight for the $\mathcal{I}$-invariant as in [48]?

8.2.2. *Triviality of the linking invariants with integral coefficients.*

The behavior of the linking invariants is not well understood. If we succeeded in solving Question 2.7, there are still open problems about them. In particular, I formulated the following conjecture in [45].

Conjecture 8.5. *The free part of the linking invariants is determined by the combinatorics.*

According to our first computations, this conjecture looks to still hold for the Artal and Rodau linking invariant. Until now, there are no mathematical arguments (even partial) that could explain this particular behavior. Understanding it could bring some light to other questions like the combinatorial nature of the characteristic varieties or the study of the branched coverings of the complex projective plane.

[19] A gap in the proof has been corrected in [10]

[20] The depth of a character $\xi \in (\mathbb{C}^*)^n$ is the maximal value of i such that $\xi \in V_i(\mathcal{A})$,

8.2.3. *Generalizations of the linking invariants.*

As previously mentioned, the loop-linking numbers are efficient topological invariants for line arrangements. As mentioned in the introduction, line arrangements are at the intersection of hyperplane arrangements and algebraic plane curves or more generally divisors in the projective surface, e.g. rational surfaces. So it could be useful to generalize the linking invariants in these two natural directions.

A generalization to hyperplane arrangements will imply an adaptation of the construction in higher dimensions. This will allow us to better understand the topology of these objects. In particular, by combining this construction with the notion of parallel connection, we could prove that the diffeomorphism type of the complement of a hyperplane arrangement does not determine its embedded topology.

In the other direction, a generalization to divisors in rational surfaces will require a more careful study of the singularities of the divisor and of the projective space. This will bring a new point of view and a new tool in the study of curves in weighted projective spaces.

8.2.4. *Connection with other invariants.*

Until now, the $\mathcal{I}$-invariant is only related to the characteristic varieties. Nevertheless, some behaviors tend to suggest the existence of connections with other invariants. The linking invariants take values in an Abelian module with torsion. The order of the torsion θ is determined by the combinatorics of the arrangements. First, it would be useful to have an efficient way to compute it. Among the examples studied, the value of θ seems to be related to the structure of the moduli space. In the Appendix of [45], lots of Zariski pairs whose moduli space is parametrized by the primitive 5th roots of unity, have been distinguished using a linking invariant for which $\theta = 5$. Moreover, it appears that all the arrangements with 11 lines that share these two properties are distinguished by the linking-invariant [45, Section 6]. The example, which proves that the invariant of Rodau and Artal is finer than the loop-linking number, also share them. A similar phenomenon also exists between θ and the Alexander invariant isomorphism test developed by Artal, Carmona, Cogolludo and Marco in [8], see [11, 45, 44] for illustrations of this phenomenon.

8.3. **Moduli space & combinatorial structures.**

To understand moduli spaces of line arrangements is a hard question. Nevertheless, due to the Randell lattice-isotopy theorem, they play a central role in the study of the topology and the geometry of line arrangements.

8.3.1. *Pathological examples.*

Recently, Core and Luber produced the first explicit example of a moduli space of line arrangement with singularities [22]. The geometry of arrangements in a continuous family passing through such a singular point has to be investigated. Are there some geometric characterizations of each irreducible component that appear at the same time for the arrangement at the singularity? In a similar direction, constructing other explicit examples of pathological moduli space could be interesting.

Problem 8.6. *Construct an explicit example of a moduli space of line arrangement with a singularity of the type* $\{xy = xz = 0\}$*, or which is not of pure dimension.*

Having a better understanding of such pathological cases could help, for example, to construct a counter-example to Terao's conjecture [67].

8.3.2. *Combinatorial structure.*

In [44], a combinatorial structure named the splitting-polygon, suggesting a non-connected moduli space is given. It allows the construction of numerous examples of Zariski pairs of a new type: the Galois group of their definition field is isomorphic to the Klein group. In a work-in-progress, a similar technique is used to construct non-arithmetic pairs, some of them being defined over the rational numbers. This splitting-polygon only enables splitting into 2 parts of the moduli space, so successive applications of this structure can only induce a moduli space with 2^k connected components. A generalization of this structure which induces a splitting in more than 2 components would be welcome. The work made in [49] would be a good starting point.

8.3.3. *κ_m arrangements.*

Following the construction made in Section 7.1, one can define complexity classes κ_m on the set of arrangements as follows. An arrangement $\mathcal{A}$ is κ_m, if m is the minimal integer such that $\mathcal{C}(\mathcal{A})$ admits a m-perturbation. For example, the class κ_0 corresponds to the inductively connected arrangements.

At first, we could investigate the class of κ_1 arrangements. It contains lots of the known examples of Zariski pairs, and it is connected to the notion of splitting-polygons [44]. When in addition their naive dimension is null, then their moduli spaces are either connected or 0-dimensional. The question is then to know if there exists a combinatorial method to determine in which case the arrangement is. For some specific arrangement, the combinatorial class of arrangements with a rigid pencil form [49] gives an answer, like in Example 6.24. Nevertheless, it doesn't work in general.

Let $\mathfrak{C}_m(n)$ be the maximal number of connected components in the moduli space of a κ_m arrangement of n lines. The example given in Section 7.4 is κ_1 and the number of connected components of its moduli space grows linearly in $n = |\mathcal{A}|$. There are combinatorial reasons to think that we could have a growth in $3n$ for κ_1 arrangements, but could it grow faster?

Problem 8.7. *For a fix $m \in \mathbb{N}$, determine the behavior of $\mathfrak{C}_m(n)$ when n growths.*

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